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Essays on forward portfolio theory and financial time series modeling

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Essays on forward portfolio theory and financial time series modeling

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This dissertation contains four self-contained essays that explore the application of stochastic and statistical modeling techniques to the problem of optimal portfolio choice and financial time series analysis.

The first essay presents turnpike-type results for the risk tolerance function in an incomplete Itô-diffusion market setting under time-monotone forward performance criteria. We show that, contrary to the classical case, the temporal and spatial limits do not coincide. Rather, we establish that they depend directly on the left- and right-end of the support of an underlying measure, used to construct the forward performance criterion. We provide examples with discrete and continuous measures, and discuss the asymptotic behavior of the risk tolerance for each case.

The second essay examines the long term behavior of the optimal wealth and optimal portfolio weights processes in an Itô-diffusion market under the

time-monotone forward performance criteria. We show that the underlying measure μ associated with the forward performance criterion defines the risk profile of the investor, and in turn determines the optimal portfolio strategy and optimal wealth in the long run.

The third essay considers two fund managers who trade under relative performance concerns, depending on each other's strategies, in an Itô-diffusion market, We analyze both the passive and the competitive cases, and under both asset specialization and diversification. To allow for dynamic model revision and flexible investment horizons, we introduce the concept of relative forward performance for the passive case, and the notion of forward Nash equilibrium for the competitive one. For homothetic forward criteria, we provide explicit solutions for all cases.

In the fourth essay, we assess the dynamics of realized betas, relative to the dynamics in the underlying market variance and covariances with the market, using 5-minute high-frequency asset prices of the DJIA component stocks from January 1, 2010 to December 31, 2014. We find that, unlike the realized variances and covariances which fluctuate widely and are highly persistence, the realized beta series, on the other hand, display much less persistence. We then construct a simple autoregressive plus noise DLM time series model for the realized beta, where the measurement error follows a normal distribution centered at zero with asymptotically valid variance given in [7]. This approach helps us obtain samples from filtered and smoothed true underlying beta series and forecast future betas.

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Chapter 1

Temporal and spatial turnpike-type results under forward time-monotone performance criteria

1.1 Introduction

Turnpike results in maximal expected utility models yield the behavior of optimal portfolio functions when the investment horizon is long, under asymptotic assumptions on the investor's risk preferences.

The essence of the "turnpike" result (stated, for simplicity, for a single log-normal stock with coefficients μ and σ) is the following: assume that the investment horizon is $[0,T]$ and that the investor's utility U_T behaves like a power function for large wealth levels, i.e.,

$$U_T\left(x\right) \sim \frac{1}{\gamma}x^{\gamma}, \qquad x \quad \text{large.} \tag{1.1}$$

Then, if this horizon is very long, the associated optimal portfolio function $\pi^*\left(x,t;T\right)$ is "close" to the one corresponding to this power utility, i.e., for *each* $x>0$, $t\in[0,T]$,

$$\frac{\pi^*\left(x,t;T\right)}{x} \sim \frac{\mu}{\sigma^2}\frac{1}{1-\gamma}, \qquad T \quad \text{large.} \tag{1.2}$$

In other words, the asymptotic *spatial* behavior of the terminal datum dictates the long-term *temporal* behavior of the portfolio function for *every* wealth level.

We recall that the function $\pi^*(x,t;T)$ is the one the determines the optimal wealth process in feedback form, in that the optimal wealth process X_t^*, $t \in [0,T]$, is generated by the investment strategy $\pi_t^* = \pi^*(X_t^*, t; T)$.

Turnpike results can be found in [20] where a continuous-time model was first considered, and the turnpike properties were established using contingent claim methods. Their results were later extended in [35] using an autonomous equation that the function $\pi(x,t;T)$ satisfies and arguments from viscosity solutions. Duality methods were used in [22] for complete markets and the incomplete market case was studied in [33].

More recently, the authors of [11] established the rate of convergence in a log-normal model, showing that there exist a positive constant c and a function $D(x)$, such that, for all $x > 0$,

$$\left| \pi^*(x,t;T) - \frac{\mu}{\sigma^2} \frac{1}{1-\gamma} x \right| \le D(x)\, e^{-c(T-t)}.$$

A closer look at these turnpike results yields that we are essentially working in a *single* investment horizon setting, $[0,T]$, which is taken to be very long. As a result, however, one needs to commit to a market model for this long horizon, but this choice cannot be modified later on, if time-consistency is desired. Furthermore, one pre-commits at initial time to a utility function for very far in the future, T. We also remark that no matter how big T is, the optimal investment problem is not defined beyond this point, because the utility function is given for T only.

Herein, we take an alternative point of view. Instead of committing to a single long horizon $[0,T]$, we define an investment problem for all times $t \in [0,\infty)$. Moreover, instead of choosing at an initial time the utility U_T for the remote horizon T, we choose the utility at this initial time. We also depart from the log-normal setting and work with a general Ito-diffusion multi-security incomplete market model.

We measure the performance of investment strategies via the so-called forward investment performance criterion. This criterion was introduced by Musiela and one of the authors in [55] and offers flexibility for performance measurement and risk management under model adaptation and ambiguity, alternative market views, rolling horizons, and others. We recall its definition and refer the reader to, among others, [57], [59], for an overview of the forward approach.

Herein, we focus on the class of time-monotone forward performance criteria, studied in [58] and briefly reviewed in the next section. They are given by a time-decreasing and adapted to the market information process, $U(x,t)$, $(x,t) \in \mathbb{R}_+ \times [0,\infty)$, of the form

$$U(x,t) = u(x, A_t),$$

where $u(x,t)$ is a deterministic function (see (1.14)) and $A_t = \int_0^t |\lambda_s|^2 \, ds$, with the process λ_t being the market price of risk. Note that $U(x,t)$ is a compilation of a deterministic investor-specific input, $u(x,t)$, and a stochastic market-specific input, A_t.

The optimal investment process π_t^* turns out to be, for $t \geq 0$,

$$\pi_t^* = \sigma_t^+ \lambda_t r\left(X_t^*, A_t\right) \qquad \text{with} \qquad r(x,t) := -\frac{u_x(x,t)}{u_{xx}(x,t)}, \tag{1.3}$$

where σ_t^+ is the pseudo-inverse of the volatility matrix, and X_t^*, $t \geq 0$, the optimal wealth generated by this investment strategy π_t^* (cf. (1.12)). The function $r(x,t)$ is the (local) risk tolerance and will be the main object of study herein.

Contrary to the classical case, in which a terminal datum is pre-assigned for T and the solution is then constructed for $t \in [0,T)$, in the forward setting, the criterion is defined for all times, starting with an initial (and not terminal) datum $u_0(x) = U(x,0)$.

In analogy to the classical turnpike setting, we are thus motivated to study the following question: if the initial condition $u_0(x)$ is such that

$$u_0(x) \sim \frac{1}{\gamma} x^{\gamma}, \qquad x \ \text{ large}, \tag{1.4}$$

does this imply that, for each $x > 0$,

$$\frac{r(x,t)}{x} \sim \frac{1}{1-\gamma}, \quad t \ \text{ large } \ ?$$

There are fundamental differences between the classical and the forward settings, for one is not a mere variation of the other by a time reversal. Rather, the classical problem is well-posed while the forward is an inverse problem.

Naturally, various properties used for the classical turnpike results fail, with the most important being the lack of comparison principle for various PDEs (cf. (1.14) and (1.22)) at hand.

The first striking difference between the two settings is the distinct nature of the temporal and spatial limits. Indeed, in the traditional turnpike results in [35] and [11], the temporal limit in (1.2) coincides with the spatial one, in that for fixed time T_0 and wealth level x_0,

$$\lim_{x\uparrow\infty}\frac{\pi\left(x,t;T_0\right)}{x}=\lim_{T\uparrow\infty}\frac{\pi\left(x_0,t;T\right)}{x_0}.$$

However, this is *not* the case in the forward setting. Indeed, the temporal and spatial limits of the function $\frac{r(x,t)}{x}$ do *not* coincide. This can be seen, for instance, in the motivational example in section 2.1.

The aim herein then becomes the study of the *spatial* and *temporal* limits

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x}\qquad\text{and}\qquad\lim_{t\uparrow\infty}\frac{r(x_0,t)}{x},\tag{1.5}$$

for fixed $t_0>0, x_0>0$, respectively, under appropriate conditions for the asymptotic behavior of the initial datum $u_0\left(x\right)$, for large x.

Pivotal role for determining these limits is played by an underlying positive finite Borel measure, μ, which is the defining element for the construction of the forward performance process. Indeed, it was shown in [58] that the above function u is uniquely (up to an additive constant) related to a harmonic function $h:\mathbb{R}\times[0,\infty)\longrightarrow\mathbb{R}^+$, and, furthermore, the latter is uniquely characterized by an integral transform, specifically,

$$u_x\left(h\left(z,t\right),t\right)=-e^{-x+\frac{t}{2}} \qquad \text{with} \qquad h\left(z,t\right)=\int_a^b e^{zy-\frac{1}{2}y^2t}\mu\left(dy\right), \tag{1.6}$$

for $0\leq a\leq b\leq\infty$.

An immediate consequence of this general solution is that the initial datum is directly related to this measure μ, in that $\left(u_0'\right)^{(-1)}$ needs to be of the integral form

$$\left(u_0'\right)^{(-1)}\left(x\right)=\int_a^b x^{-y}\mu\left(dy\right).$$

As a result, it is natural to expect that the asymptotic properties of $u_0\left(x\right)$, which enter in the turnpike assumptions, are also directly linked to the form and properties of μ.

Furthermore, this measure also appears in the specification of the risk tolerance function. Indeed, we deduce from (1.3) and (1.6) that $r\left(x,t\right)$ can be represented as

$$r\left(x,t\right)=h_x\left(h^{(-1)}\left(x,t\right),t\right), \tag{1.7}$$

with both h_x and $h^{(-1)}$ depending on μ.

The main results herein are that, if the support of the measure is finite, $b<\infty$, then the *spatial* limit coincides with the *right-end* point of the support while the *temporal* limit with the *left-end* one, namely,

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x}=b \qquad \text{and} \qquad \lim_{t\uparrow\infty}\frac{r(x_0,t)}{x}=a. \tag{1.8}$$

The first step in obtaining the above limits is to understand the connection between the asymptotic behavior of the initial (marginal) datum and the finiteness of the measure's support. We study the following two cases, which correspond to the spatial and temporal limits, respectively.

We first show that the asymptotic assumption (1.4), stated in terms of the marginal,

$$u_0'(x) \quad \sim \quad x^{\gamma-1}, \tag{1.9}$$

if and only if the right end of the measure's support satisfies both $b = \frac{1}{1-\gamma}$ and $\mu(\{b\}) = 1$. In other words, condition (1.9) implies that the measure must have finite support with its right boundary equal to $\frac{1}{1-\gamma}$ and, furthermore, with a mass at this point. Conversely, for the measure to have these properties, condition (1.9) must hold. We then establish the first limit in (1.8) using representation (1.6), the equation (1.14) satisfied by $u(x,t)$, and various convexity properties of h and its derivatives. We stress that the requirement that $\mu(\{b\}) \neq 0$ cannot be relaxed. Indeed, we show in Example 6.2, where the measure is the Lebesgue one, that the spatial turnpike property fails.

For the second case, we relate the finiteness of the measure's support with a relaxed version of (1.9). We show that if there exists $\gamma < 1$, $\gamma \neq 0$, such that for all $\gamma' \in (\gamma, 1)$ and $\gamma'' < \gamma$,

$$\lim_{x\uparrow\infty} \frac{u_0'(x)}{x^{\gamma'-1}} = 0 \quad \text{and} \quad \lim_{x\uparrow\infty} \frac{u_0'(x)}{x^{\gamma''-1}} = \infty, \tag{1.10}$$

then the right boundary of the measure's support must satisfy $b = \frac{1}{1-\gamma}$, and vice-versa. This regular variation assumption is weaker than (1.9), needed for the spatial limit and, naturally, yields a weaker result. Indeed, while the support has to be finite with right boundary equal to $\frac{1}{1-\gamma}$, it does not need to have a mass at $\frac{1}{1-\gamma}$.

We in turn establish the second limit in (1.8), which is the genuine analogue of the classical turnpike result. Obtaining this limit is considerably more challenging than in the classical case due to the ill-posed nature of the problem. Indeed, the methodology used in [35] is inapplicable because of lack of comparison results for the ergodic version of the equation satisfied by $r(x,t)$. The approach of [11] does not apply either because of the lack of connection between the solutions of the ill-posed heat equation and Feynman-Kac type stochastic representation of its solution. Therefore, one needs to work directly with the function $r(x,t)$, which, from (1.7) and (1.6), is given in the implicit form

$$r(x,t) = \int_a^b y e^{y h^{(-1)}(x,t) - \frac{1}{2}y^2 t} \mu(dy),$$

where however the spatial inverse $h^{(-1)}$ is involved.

The key step in obtaining the temporal limit is to show that

$$\lim_{t\uparrow\infty} \frac{h^{(-1)}(x,t)}{t} = \frac{a}{2},$$

where a is the left end point of the measure's support. Then the temporal convergence in (1.8) and the rate of convergence is shown using the implicit representation

$$r(x,t) - ax = \int_a^b (y-a)\, e^{ty\left(\frac{h^{(-1)}(x,t)}{t} - \frac{1}{2}y\right)} \mu(dy).$$

In addition to the general spatial and temporal convergence results, we present two representative examples. In the first, the measure is a finite sum of Dirac functions while, in the second, it is taken to be the Lebesgue measure. We calculate the limits of (1.8), and also provide asymptotic expansions for the risk tolerance function.

The paper is structured as follows. In section 2, we present the market model, the investment performance criterion and a motivating example demonstrating that the temporal and spatial limits do not in general coincide. In sections 3 and 4, we analyze respectively the spatial and temporal asymptotic behavior of the relative risk tolerance, while in section 5 we analyze the asymptotic properties of the relative prudence function. In section 6 we present the two representative examples, and conclude in section 7 with future research directions.

1.2 The model and the investment performance criterion

The market environment consists of one riskless and k risky securities. The prices of the risky securities are modelled as Itô processes, namely, the price S^i of the i^{th} risky asset follows

$$dS_t^i = S_t^i \left(\mu_t^i dt + \Sigma_{j=1}^d \sigma_t^{ji} dW_t^j \right),$$

with $S_0^i > 0$, for $i = 1, ..., k$. The process $W_t = \left(W_t^1, ..., W_t^d \right)$, $t \geq 0$, is a standard Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The coefficients μ_t^i and $\sigma_t^i = \left(\sigma^{1i}, ..., \sigma_t^{di} \right)$, $i = 1, ..., k$, $t \geq 0$, are $\mathcal{F}_t-$adapted processes and values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. We denote by σ_t the volatility matrix, i.e. the $d \times k$ random matrix $\left(\sigma_t^{ji}\right)$, whose i^{th} column represents the volatility σ_t^i of the i^{th} asset. We may, then, alternatively, write the above equation as

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i \cdot dW_t\right).$$

The riskless asset, the savings account, has price process B satisfying $dB_t = r_t B_t dt$ with $B_0 = 1$, and for a nonnegative $\mathcal{F}_t-$adapted interest rate process r_t. Also, we denote by μ_t the k-dimensional vector with coordinates μ_t^i and by $\mathbf{1}$ the k-dim vector with every component equal to one. The processes μ_t, σ_t and r_t satisfy the appropriate integrability conditions.

We assume that $\mu_t - r_t\mathbf{1} \in Lin\left(\sigma_t^T\right)$, where $Lin\left(\sigma_t^T\right)$ denotes the linear space generated by the columns of σ_t^T. Therefore, the equation $\sigma_t^T z = \mu_t - r_t\mathbf{1}$ has a solution, known as the market price of risk,

$$\lambda_t = \left(\sigma_t^T\right)^+ \left(\mu_t - r_t\mathbf{1}\right). \tag{1.11}$$

It is assumed that there exists a deterministic constant $c > 0$, such that $|\lambda_t| \leq c$ and that $\lim_{t\uparrow\infty} \int_0^t |\lambda_s|^2 ds = \infty$.

Starting at $t = 0$ with an initial endowment $x \geq 0$, the investor invests at any time $t > 0$ in the risky and riskless assets. The present value of the amounts invested are denoted by the processes π_t^0 and π_t^i, $i = 1, ..., k$, respectively, which are taken to be self-financing. The present value of her investment is then given by the discounted wealth process $X_t^\pi = \sum \pi_t^i$, $t > 0$, which solves

$$dX_t^\pi = \sigma_t \pi_t \cdot \left(\lambda_t dt + dW_t\right) \tag{1.12}$$

with the (column) vector $\pi_t = (\pi_t^i; i = 1, ..., k)$. It is taken to satisfy the non-negativity constraint $X_t^\pi \geq 0$, $t > 0$.

The set of admissible policies is given by

$$\mathcal{A} = \left\{ \pi : \text{self-financing}, \ \ \pi_t \in \mathcal{F}_t, \ \ E_{\mathbb{P}} \int_0^t |\sigma_s \pi_s|^2 \, ds < \infty, \ \ X_t^\pi \geq 0, \ \ t > 0 \right\}.$$

The performance of admissible investment strategies is evaluated via the so-called forward investment performance criteria, introduced in [55] (see, also [56], [57] and [59]). We review their definition next.

We introduce the domain notation $\mathbb{D}_+ = \mathbb{R}_+ \times [0, \infty)$ and $\mathbb{D} = \mathbb{R} \times [0, \infty)$.

Definition 1.2.1. *An $\mathcal{F}_t$-adapted process $U(x,t)$ is a forward investment performance if for $(x,t) \in \mathbb{D}$,*

i) the mapping $x \to U(x,t)$ is strictly increasing and strictly concave;

ii) for each $\pi \in \mathcal{A}$, $E_{\mathbb{P}}(U(X_t^\pi, t))^+ < \infty$, and for $s \geq t$,

$$U\left(X_t^\pi, t\right) \geq E_{\mathbb{P}}\left(U(X_s^\pi, s) \middle| \mathcal{F}_t\right),$$

iii) there exists $\pi^* \in \mathcal{A}$ such that for $s \geq t$,

$$U\left(X_t^{\pi^*}, t\right) = E_{\mathbb{P}}\left(U(X_s^{\pi^*}, s) \middle| \mathcal{F}_t\right).$$

Herein we focus on the class of *time-monotone* forward performance processes. For the reader's convenience, we rewrite some of the results we stated in the introduction. Time-monotone forward processes were extensively studied in [58], and are given by

$$U(x,t) = u(x, A_t), \tag{1.13}$$

where $u : \mathbb{D}_+ \to \mathbb{R}_+$ is strictly increasing and strictly concave in x, satisfying

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}}. \tag{1.14}$$

The market input processes A_t and M_t, $t \geq 0$, are defined as

$$M_t = \int_0^t \lambda_s \cdot dW_s \quad \text{ and } \quad A_t = \int_0^t |\lambda_s|^2 \, ds = \langle M \rangle_t \, . \tag{1.15}$$

The optimal portfolio process π_t^* is given by $\pi_t^* = \sigma_t^+ \lambda_t r(X_t^*, A_t)$, where the (local) risk tolerance function $r\,(x,t) : \mathbb{D}_+ \to \mathbb{R}_+$ is defined as

$$r\,(x,t) := -\frac{u_x\,(x,t)}{u_{xx}\,(x,t)}. \tag{1.16}$$

Central role in the construction of the performance criterion, the optimal policies and their wealth plays a harmonic function $h : \mathbb{D} \to \mathbb{R}_+$, defined via the transformation

$$u_x(h(z,t),t) = e^{-z+\frac{t}{2}}. \tag{1.17}$$

It solves, as it follows from (1.14) and (1.17), the ill-posed heat equation

$$h_t + \frac{1}{2} h_{zz} = 0. \tag{1.18}$$

Moreover, it is positive and strictly increasing in z. It was shown in [58], that such solutions are *uniquely* represented by

$$h(z,t) = \int_a^b \frac{e^{yz - \frac{1}{2}y^2 t} - 1}{y} \nu(dy) + C,$$

where $a=0^{+}$ or $a>0, b\leq\infty$ and C a generic constant.

The measure ν is defined on $\mathcal{B}^{+}(\mathbb{R})$, the set of positive Borel measures, with the additional properties that, for $z\in\mathbb{R}$, $\int_a^b e^{yz}\nu(dy)<\infty$ and $\int_a^b \frac{\nu(dy)}{y}<\infty$. To simplify the presentation and without loss of generality, we choose $C:=\int_a^b \frac{1}{y}\nu(dy)$ and, also, introduce the normalized measure $\mu\left(dy\right)=\frac{1}{y}\nu(dy)$.

Then, the function h has, for $(z,t)\in\mathbb{D}$, the representation

$$h(z,t)=\int_a^b e^{yz-\frac{1}{2}y^2t}\mu(dy), \tag{1.19}$$

with $\int_a^b ye^{yz}\mu(dy)<\infty,\quad a=0^{+},\, a>0,\, b\leq\infty.$

We easily deduce that for each $t_0\geq 0$, the function $h\left(.,t_0\right)$ is absolutely monotonic, since $\partial^i h\left(z,t_0\right)/\partial z^i>0$, $i=1,2...$ Such functions satisfy, for each $t_0\geq 0$, $i=1,2,...$, the inequality

$$\frac{\partial^{i+1}h\left(z,t_0\right)}{\partial z^{i+1}}\frac{\partial^{i-1}h\left(z,t_0\right)}{\partial z^{i-1}}-\left(\frac{\partial^{i}h\left(z,t_0\right)}{\partial z^{i}}\right)^2>0. \tag{1.20}$$

From (1.17), (1.16) and (1.19), we obtain that the risk tolerance function is represented as

$$r(x,t)=h_z\left(h^{(-1)}(x,t),t\right)=\int_a^b ye^{yh^{(-1)}(x,t)-\frac{1}{2}y^2t}\mu(dy). \tag{1.21}$$

Furthermore, the first equality together with (1.18) yields that it satisfies the (ill-posed) non-linear equation

$$r_t+\frac{1}{2}r^2r_{xx}=0, \tag{1.22}$$

with $r(x,0)=\int_a^b ye^{yh^{(-1)}(x,0)}\mu(dy)$.

We also have that

$$r_x(x,t) = \frac{h_{zz}\left(h^{(-1)}(x,t),t\right)}{r\left(x,t\right)} = \frac{1}{r\left(x,t\right)}\int_a^b y^2 e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2 t}\mu(dy) > 0. \quad (1.23)$$

Furthermore,

$$r_{xx}\left(x,t\right) = \frac{1}{r^3\left(x,t\right)}\left(h_{zzz}\left(z,t\right)h_z\left(z,t\right) - h_{zz}\left(z,t\right)^2\right)\Big|_{z=h^{(-1)}(x,t)} > 0, \quad (1.24)$$

where we used (1.20).

We note that we will frequently differentiate under the integral sign in (1.19), which is permitted as explained in [58]. It can be also seen directly since, after differentiation, one can show that the relevant integrands are jointly continuous in their respective arguments and thus uniformly locally integrable. This allows us to differentiate under the integral sign (see, for example, Theorem 24.5 in [3] and the remarks following it).

As stated in the introduction, the aim herein is to investigate the spatial and temporal limits in (1.5), with $r\left(x,t\right)$ as in (2.13) when the measure has *finite* support. We first provide an example which shows that, contrary to the classical case, these two limits do *not in* general coincide.

1.2.1 A motivating example

Let the underlying measure μ be a Dirac function at $\frac{1}{1-\gamma}$, $\gamma < 1$. From (1.19) and (1.17) we have that, for $t \geq 0$,

$$h(x,t) = e^{\frac{1}{1-\gamma}x - \frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t} \quad \text{and} \quad u_x(x,t) = x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t}.$$

Therefore, the local risk tolerance function is given by $r(x,t) = \frac{1}{1-\gamma}x$ and thus the spatial and temporal limits coincide,

$$\lim_{x\uparrow\infty} \frac{r(x,t_0)}{x} = \frac{1}{1-\gamma} \quad \text{and} \quad \lim_{t\uparrow\infty} \frac{r(x_0,t)}{x_0} = \frac{1}{1-\gamma},$$

for fixed t_0, x_0 respectively.

Next, let the measure μ be the sum of two Dirac functions at points $a = \frac{1}{1-\theta}$ and $b = \frac{1}{1-\gamma}$ such that $b = 2a$, with $0 < \theta < 1$ and $\gamma < 1$, i.e.,

$$\mu = \delta_{\frac{1}{1-\theta}} + \delta_{\frac{1}{1-\gamma}} \qquad \text{with} \qquad \frac{1}{1-\gamma} = 2\frac{1}{1-\theta}. \tag{1.25}$$

Then, (1.19) and (1.17) yield that $h(x,0) = e^{\frac{1}{1-\theta}x} + e^{\frac{1}{1-\gamma}x}$,

$$u_x(x,0) = 2^{1-\theta}\left(\sqrt{1+4x}-1\right)^{\theta-1} \quad \text{and} \quad u_x^{(-1)}(x,0) = x^{-\frac{1}{1-\theta}} + x^{-\frac{1}{1-\gamma}}. \tag{1.26}$$

In turn,

$$\lim_{x\uparrow\infty} \frac{u_x(x,0)}{x^{\gamma-1}} = \lim_{x\uparrow\infty} \frac{2^{2(1-\gamma)}\left(\sqrt{1+4x}-1\right)^{2(\gamma-1)}}{x^{\gamma-1}} = 1. \tag{1.27}$$

Moreover, expression (1.19) gives, for $t > 0$,

$$h(x,t) = e^{\frac{1}{1-\theta}x - \frac{1}{2}\frac{1}{(1-\theta)^2}t} + e^{\frac{2}{1-\theta}x - \frac{1}{2}\frac{2}{(1-\theta)^2}t},$$

and, thus,

$$h^{(-1)}(x,t) = \frac{1}{1-\theta}t + (1-\theta)\ln\left(\frac{\sqrt{e^{\left(\frac{1}{1-\theta}\right)^2 t} + 4x} - \sqrt{e^{\left(\frac{1}{1-\theta}\right)^2 t}}}{2}\right). \tag{1.28}$$

In turn, transformation (1.17) yields

$$u_x(x,t) = 2^{1-\theta}e^{\left(\frac{1}{2}-\frac{1}{1-\theta}\right)t}\left(\sqrt{e^{\left(\frac{1}{1-\theta}\right)^2 t} + 4x} - \sqrt{e^{\left(\frac{1}{1-\theta}\right)^2 t}}\right)^{\gamma-1}.$$

Differentiating the above to obtain $u_{xx}(x,t)$ (or using (1.19), (1.28) and (2.13)), we deduce that the risk tolerance function is given by

$$r(x,t)=\frac{x}{1-\gamma}\frac{\sqrt{4x+e^{\left(\frac{1}{1-\theta}\right)^{2}t}}}{\sqrt{e^{(\frac{1}{1-\theta})^{2}t}+4x}+\sqrt{e^{(\frac{1}{1-\theta})^{2}t}}}. \tag{1.29}$$

Therefore, for each $t_0\geq 0$,

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x}=\frac{2}{1-\theta}=\frac{1}{1-\gamma}. \tag{1.30}$$

while, for each $x_0>0$,

$$\lim_{t\uparrow\infty}\frac{r(x_0,t)}{x_0}=\frac{1}{1-\theta}. \tag{1.31}$$

Therefore, the spatial and temporal limits do *not* coincide.

Next, we make the following two important observations. Firstly, note that (1.25) yields that the support of the measure is

$$supp\ (\mu)=\left\{\frac{1}{1-\theta},\frac{1}{1-\gamma}\right\}.$$

Therefore, the *spatial* limit coincides with the *right-end* of the support while the *temporal* limit with the *left-end* one.

Secondly, for each $x_0>0$ the temporal limit of the ratio $\frac{h^{(-1)}(x_0,t)}{t}$ is equal to *half* of the *left-end* point, since (1.28) yields

$$\lim_{t\uparrow\infty}\frac{h^{(-1)}(x_0,t)}{t}$$

$$=\lim_{t\uparrow\infty}\left(\frac{1}{1-\theta}+\frac{1-\theta}{t}\ln\left(\frac{1}{2}\left(\sqrt{e^{(\frac{1}{1-\theta})^{2}t}+4x}-\sqrt{e^{(\frac{1}{1-\theta})^{2}t}}\right)\right)\right)=\frac{1}{2\left(1-\theta\right)}.$$

In section 4 we will show that these two properties are always valid. In particular, we will see that it is the limit of the above ratio that plays the key role in establishing the temporal turnpike limit for general measures.

To juxtapose the above results with the ones in the traditional expected terminal utility setting, we compute the analogous quantities and associated limits for the cases analyzed in [35] and [11] for log-normal markets. Without loss of generality, we consider a market with a riskless bond of zero interest rate and a single log-normal stock with mean rate of return μ and volatility σ.

To this end, we fix an arbitrary horizon $T > 0$ and, in analogy to (1.26), we take the *terminal* inverse marginal utility, $I(x) = (U')^{(-1)}(x)$, to be

$$I(x) = x^{-\frac{1}{1-\theta}} + x^{-\frac{1}{1-\gamma}},$$

for $x > 0$ and θ, γ as in (1.25). This corresponds to terminal marginal utility $U'(x) = \left(\frac{\sqrt{1+4x}-1}{2}\right)^{\theta-1}$ and, thus, in analogy to (1.27),

$$\lim_{x\uparrow\infty} \frac{U'(x)}{x^{\gamma-1}} = 1.$$

We now consider the value function, say $u(x,t;T)$ of the associated Merton problem, for $t \in [0,T]$. Letting $\tau = T - t$ be the time to the end of the investment horizon, we deduce, using well known results, that the function $\tilde{u}(x,\tau) \equiv u(x, T-t; T)$, satisfies, for $(x,\tau) \in \mathbb{R}_+ \times [0,T)$, the Hamilton-Jacobi-Bellman equation

$$\tilde{u}_\tau + \frac{1}{2}\lambda^2 \frac{\tilde{u}_x^2}{\tilde{u}_{xx}} = 0.$$

The inverse spatial marginal value function $\tilde{v}:\mathbb{R}_{+}\times[0,T)\rightarrow\mathbb{R}_{+}$ then solves

$$\tilde{v}_{\tau}=\frac{1}{2}\lambda^{2}x^{2}\tilde{v}_{xx}+\lambda^{2}x\tilde{v}_{x},$$

with $\tilde{v}(x,0)=I\left(x\right)$. We easily deduce that

$$\tilde{v}(x,\tau)=e^{\alpha\tau}x^{-\alpha}+e^{\beta\tau}x^{-2\alpha},$$

with $\alpha=\frac{1}{2}\lambda^{2}\frac{\theta}{(1-\theta)^{2}}$ and $\beta=\lambda^{2}\frac{1+\theta}{(1-\theta)^{2}}$. Note that $\beta>2\alpha$.

Taking the spatial inverse of $\tilde{v}(x,\tau)$ yields

$$\tilde{u}_{x}\left(x,\tau\right)=\left(\frac{e^{\alpha\tau}+\sqrt{e^{2\alpha\tau}+4xe^{\beta\tau}}}{2x}\right)^{1-\theta}.$$

Therefore, the associated risk tolerance function is given by

$$\tilde{r}(x,\tau)=\frac{1}{1-\theta}\left(\frac{2x}{1+\sqrt{1+4xe^{(\beta-2\alpha)\tau}}}+\frac{8x^{2}}{\left(\sqrt{e^{(2\alpha-\beta)\tau}}+\sqrt{e^{(2\alpha-\beta)\tau}+4x}\right)^{2}}\right).$$

In turn, for each $\tau_{0}>0$ and $x_{0}>0$, we obtain, respectively, the spatial and the temporal limits,

$$\lim_{x\uparrow\infty}\frac{\tilde{r}(x,\tau_{0})}{x}=\frac{1}{1-\theta}\qquad\text{and}\qquad\lim_{\tau\uparrow\infty}\frac{\tilde{r}(x_{0},\tau)}{x_{0}}=\frac{1}{1-\theta}.$$

1.3 Spatial asymptotic results

We examine the spatial asymptotic behavior of the risk tolerance function as $x\uparrow\infty$, for each $t_{0}\geq 0$, under asymptotic assumptions for large wealth levels of the investor's initial risk preferences. In accordance with similar assumptions in [35] and [11], we impose this asymptotic assumption on the marginal $u_0'(x)$ instead of the function itself.

Assumption 1: *The initial datum* u_0 satisfies, for some $\gamma < 1$,

$$\lim_{x \uparrow \infty} \frac{u_0'(x)}{x^{\gamma-1}} = 1. \tag{1.32}$$

The next result yields necessary and sufficient conditions on b, the right end of the support of the measure, for the above assumption to hold.

Lemma 1.3.1. *Assumption (1.32) holds if and only if the associated measure μ satisfies*

$$b = \frac{1}{1-\gamma} \qquad \text{and} \quad \mu\left(\left\{\frac{1}{1-\gamma}\right\}\right) = 1. \tag{1.33}$$

Proof. From (1.32), (1.17) and the fact that $h(x,0)$ is strictly increasing and of full range, we have

$$1 = \lim_{x \uparrow \infty} \frac{u_x(x,0)}{x^{\gamma-1}} = \lim_{z \uparrow \infty} \frac{u_x(h(z,0),0)}{(h(z,0))^{\gamma-1}} = \lim_{z \uparrow \infty} \left(\frac{h(z,0)}{e^{\frac{1}{1-\gamma}z}}\right)^{1-\gamma}. \tag{1.34}$$

Therefore, representation (1.19) gives

$$\lim_{z \uparrow \infty} \int_a^b e^{z(y-\frac{1}{1-\gamma})}\mu(dy) = 1. \tag{1.35}$$

If $a = b$, then (1.33) follows directly. If $a < b$, then, it must be that $a \leq \frac{1}{1-\gamma}$, otherwise, we get a contradiction. In turn, for $\varepsilon > 0$,

$$\int_a^b e^{z(y-\frac{1}{1-\gamma})}\mu(dy) \geq \int_{\frac{1}{1-\gamma}+\varepsilon}^b e^{z(y-\frac{1}{1-\gamma})}\mu(dy) \geq e^{\varepsilon z}\mu\left([\frac{1}{1-\gamma}+\varepsilon, b]\right). \tag{1.36}$$

Sending $\varepsilon \downarrow 0$ and using (1.35) yield that $\mu((\frac{1}{1-\gamma}, b]) = 0$, and thus, $\mathrm{supp}(\mu) \subseteq (a, \frac{1}{1-\gamma}]$. Moreover, we have from (1.35) that

$$1 = \lim_{z \uparrow \infty} \int_a^{\left(\frac{1}{1-\gamma}\right)^-} e^{z(y - \frac{1}{1-\gamma})} \mu(dy) + \mu(\{\frac{1}{1-\gamma}\}) = \mu(\{\frac{1}{1-\gamma}\}),$$

and we conclude. The rest of the proof follows easily. $\square$

We next state the main spatial asymptotic result.

Proposition 1.3.1. *Suppose that the initial datum u_0 satisfies the asymptotic property (1.32). Then, for each $t_0 \geq 0$, the relative risk tolerance converges to the right-end of the support of the measure μ,*

$$\lim_{x \uparrow \infty} \frac{r(x, t_0)}{x} = \frac{1}{1-\gamma}. \tag{1.37}$$

Proof. Let $t_0 \geq 0$. From representation (1.36) we have that

$$h(z, t_0) = \int_a^{\left(\frac{1}{1-\gamma}\right)^-} e^{zy - \frac{1}{2} t_0 y^2} \mu(dy) + e^{\frac{1}{1-\gamma} z - \frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0},$$

and, in turn, the dominated convergence theorem implies

$$\lim_{z \uparrow \infty} \frac{h(z, t_0)}{e^{\frac{1}{1-\gamma} z - \frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0}} = 1. \tag{1.38}$$

Therefore, from (1.17), together with the strict monotonicity and full range of $h(z, t_0)$, we deduce that

$$\lim_{x \uparrow \infty} \frac{u_x(x, t_0)}{x^{\gamma - 1} e^{-\frac{\gamma}{2(1-\gamma)} t_0}} = 1, \tag{1.39}$$

since

$$\lim_{x \uparrow \infty} \frac{u_x(x, t_0)}{x^{\gamma - 1} e^{-\frac{\gamma}{2(1-\gamma)} t_0}} = \lim_{z \uparrow \infty} \frac{e^{-z + \frac{t_0}{2}}}{h^{\gamma - 1}(z, t_0) e^{-\frac{\gamma}{2(1-\gamma)} t_0}}$$

$$= \lim_{z\uparrow\infty} \left(\frac{h(z,t_0)}{e^{\frac{1}{1-\gamma}z - \frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0}} \right)^{1-\gamma} = 1.$$

Next, we claim that

$$\lim_{x\uparrow\infty} \frac{u_{xx}(x,t_0)}{x^{\gamma-2}e^{-\frac{\gamma}{2(1-\gamma)}t_0}} = \frac{1}{\gamma-1}. \tag{1.40}$$

To prove this, it suffices to show that for any $t_0 \geq 0$, $u_x(x,t_0)$ is convex since the above would then follow from the arguments in Lemma 3.1 (ii) in [35]. To this end, differentiating (1.17) yields

$$u_{xxx}\left(h(z,t_0),t_0\right)\left(h_z(z,t_0)\right)^2 + u_{xx}(h(z,t_0),t_0)h_{zz}(z,t_0) = e^{-z+\frac{t_0}{2}}. \tag{1.41}$$

The strict convexity of h and the strict concavity of u then give

$$u_{xxx}\left(h(z,t_0),t_0\right) > 0, \tag{1.42}$$

and using the strict monotonicity and full range of h we conclude.

Combining (1.39) and (1.40) yields

$$\lim_{x\uparrow\infty} \frac{r(x,t_0)}{x} = \lim_{x\uparrow\infty} \left(-\frac{u_x(x,t_0)}{xu_{xx}(x,t_0)} \right)$$

$$= \lim_{x\uparrow\infty} \left(-\frac{u_x(x,t_0)}{x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t_0}} \left(\frac{u_{xx}(x,t_0)}{x^{\gamma-2}e^{-\frac{\gamma}{2(1-\gamma)}t_0}} \right)^{-1} \right) = \frac{1}{1-\gamma}.$$

$\square$

We stress that assumption (1.32), or equivalently (1.33), *cannot* be weakened. Indeed, as we will see in example 6.2, where we take the measure to be the Lebesgue on $[a,b]$, and thus there is no mass at b, the spatial turnpike property does not hold.

Corollary 1.3.1. *Suppose that the initial datum u_0 satisfies the asymptotic property (1.32). Then, for each $t_0 \geq 0$,*

$$\lim_{x \uparrow \infty} r_x (x, t_0) = \frac{1}{1-\gamma}. \tag{1.43}$$

Proof. From (1.24) we have that, for each $t_0 \geq 0$, $\lim_{x \uparrow \infty} r_x (x, t_0)$ exists, and we easily conclude. □

1.4 Temporal (turnpike) asymptotic results

We investigate the temporal asymptotic behavior of the relative risk tolerance as $t \uparrow \infty$, for each $x_0 > 0$, under asymptotic assumption of the initial marginal utility for large wealth levels. This is the genuine "turnpike" analogue of similar results in classical expected utility models and the main finding herein. It shows that the relative risk tolerance will converge to the *left-end* of the support of the underlying measure μ.

As in the spatial case, we first relate the properties of the measure to the asymptotic behavior of the initial (marginal) datum.

Assumption 2: *There exists $\gamma < 1$ such that for all $\gamma' \in (\gamma, 1)$,*

$$\lim_{x \uparrow \infty} \frac{u_0' (x)}{x^{\gamma' - 1}} = 0, \tag{1.44}$$

while, for all $\gamma'' < \gamma$,

$$\lim_{x \uparrow \infty} \frac{u_0' (x)}{x^{\gamma'' - 1}} = \infty. \tag{1.45}$$

As we show next, the above assumption is directly related to a condition introduced in [36] and [22], for a discrete and a continuous-time case, respectively.

Lemma 1.4.1. *Assumption 2 is equivalent to the function* $u_0'(x)$ *varying regularly at infinity with exponent* $\gamma - 1$*, i.e. for all* $k > 0$*,*

$$\lim_{x \uparrow \infty} \frac{u_0'(kx)}{u_0'(x)} = k^{\gamma-1}. \tag{1.46}$$

Proof. We first show that condition (1.46) implies (1.44) and (1.45). We argue by contradiction. Suppose that (1.44) does not hold. Then, there exists $\gamma' \in (\gamma, 1)$ and $\varepsilon > 0$ such that for x large enough, $\frac{u_0'(x)}{x^{\gamma'-1}} > \varepsilon$. On the other hand, condition (1.46) implies that, for all $k > 0$ and x large enough, $\left|\frac{u_0'(kx)}{u_0'(x)k^{\gamma-1}} - 1\right| < \varepsilon$. Thus, for large enough x,

$$0 < \frac{u_0'(kx)}{(kx)^{\gamma'-1}} = \frac{u_0'(kx)}{u_0'(x)k^{\gamma-1}} \frac{u_0'(x)}{x^{\gamma'-1}} k^{\gamma-\gamma'} < (1+\varepsilon)\frac{u_0'(x)}{x^{\gamma'-1}} k^{\gamma-\gamma'}.$$

Since $\gamma - \gamma' < 0$, $\lim_{k\uparrow\infty} \frac{u_0'(kx)}{(kx)^{\gamma'-1}} = 0$, and we conclude. Working similarly, we establish (1.45).

Next, we show the reverse direction. Assume that (1.45) and (1.44) hold. Then, for all $\delta, k > 0$ and x large enough,

$$\frac{u_0'(kx)}{(kx)^{\gamma+\delta-1}} < 1 \qquad \text{and} \qquad \frac{x^{\gamma-\delta-1}}{u_0'(x)} < 1.$$

Multiplying these two equations and rearranging gives, for all $\delta > 0$,

$$\frac{u_0'(kx)}{u_0'(x)} < \frac{(kx)^{\gamma+\delta-1}}{x^{\gamma-\delta-1}} = k^{\gamma+\delta-1}x^{2\delta}.$$

Similarly, it follows from interchanging kx and x in the above two inequalities that

$$\frac{u_0'(kx)}{u_0'(x)} > \frac{(kx)^{\gamma-\delta-1}}{x^{\gamma+\delta-1}} = k^{\gamma-\delta-1}x^{-2\delta},$$

and condition (1.46) follows by sending first $\delta \downarrow 0$ and then $x \uparrow \infty$. $\square$

Assumption 2 is weaker than Assumption 1, and implies, as we show next, that the measure μ has support with right-end point at $\frac{1}{1-\gamma}$, but without necessarily having a mass therein.

Lemma 1.4.2. *Assumption 2 holds if and only if the measure μ has finite support with its right boundary at $\frac{1}{1-\gamma}$, namely,*

$$\inf\left\{y > 0 : \mu\left((y,\infty)\right) = 0\right\} = \frac{1}{1-\gamma}. \tag{1.47}$$

Proof. We show that Assumption 2 implies property (1.47). For each $\gamma' \in (\gamma, 1)$, we deduce from (1.44) that

$$0 = \lim_{x\uparrow\infty} \frac{u_x(x,0)}{x^{\gamma'-1}} = \lim_{z\uparrow\infty} \frac{u_x(h(z,0),0)}{(h(z,0))^{\gamma'-1}} = \lim_{z\uparrow\infty}\left(\frac{h(z,0)}{e^{\frac{z}{1-\gamma'}}}\right)^{1-\gamma'},$$

and, thus,

$$\lim_{z\uparrow\infty}\int_a^b e^{z\left(y-\frac{1}{1-\gamma'}\right)}\mu(dy) = 0. \tag{1.48}$$

Next, observe that if $b \geq 1$, then it will contradict the above limit, and thus we need to have $b < 1$. Assume now that there exists $\gamma' \in (\gamma, 1)$ with $b = \frac{1}{1-\gamma'}$. Then, for each $\tilde{\gamma} \in (\gamma, \gamma')$ we have $\frac{1}{1-\tilde{\gamma}} < \frac{1}{1-\gamma'}$ and the above gives, for ε small enough,

$$\lim_{z\uparrow\infty}\left(\int_a^{\left(\frac{1}{1-\tilde{\gamma}}+\varepsilon\right)^-} e^{z\left(y-\frac{1}{1-\tilde{\gamma}}\right)}\mu\left(dy\right)+\int_{\frac{1}{1-\tilde{\gamma}}+\varepsilon}^{b} e^{z\left(y-\frac{1}{1-\tilde{\gamma}}\right)}\mu\left(dy\right)\right)=0.$$

Therefore, it must be that $\mu\left([\frac{1}{1-\tilde{\gamma}}+\varepsilon,b]\right)=0$. Sending $\varepsilon\downarrow 0$, gives $\mu\left(\left(\frac{1}{1-\tilde{\gamma}},b\right]\right)=0$, which is a contradiction. Thus, we must have $b\leq\frac{1}{1-\gamma}$. Similarly, using (1.45) we obtain that $b\geq\frac{1}{1-\gamma}$, and, thus, $b=\frac{1}{1-\gamma}$.

To show the reverse direction, we first observe that property (1.47) and the dominated convergence theorem yield that, for any $\varepsilon>0$,

$$\lim_{z\uparrow\infty}h(z,0)e^{-(\frac{1}{1-\gamma}+\varepsilon)z}=\lim_{z\uparrow\infty}\int_a^{\frac{1}{1-\gamma}} e^{z\left(y-\left(\frac{1}{1-\gamma}+\varepsilon\right)\right)}\mu(dy)=0.$$

Then, setting γ' such that $\frac{1}{1-\gamma'}=\frac{1}{1-\gamma}+\varepsilon$, we deduce (1.44) for all $\gamma'\in(\gamma,1)$.

The rest of the proof follows easily and it is thus omitted. $\square$

We have so far established that under Assumption 2 the associated measure μ has a finite right boundary (but not necessarily a mass) at $\frac{1}{1-\gamma}$, and vice-versa.

We now turn our attention to the left boundary of the support, denoted by a, where

$$a:=\inf\{y\geq 0:\mu\left((0,y]\right)>0\}. \tag{1.49}$$

In the upcoming proofs we will frequently use the identity

$$x_0=\int_a^{\frac{1}{1-\gamma}} e^{yh^{(-1)}(x_0,t)-\frac{1}{2}y^2t}\mu(dy), \tag{1.50}$$

for $x_0 > 0$, which follows directly from (1.19) for $b = \frac{1}{1-\gamma}$.

Lemma 1.4.3. *Let $h^{(-1)} : \mathbb{D}_+ \to \mathbb{R}$ be the spatial inverse of h, and a as in (1.49). Then, for each $x_0 > 0$, $\lim_{t\uparrow\infty} h_t^{(-1)}(x_0, t)$ exists and, moreover, for $t \geq 0$,*

$$\frac{a}{2} \leq h_t^{(-1)}(x_0, t) \leq \frac{1}{2(1-\gamma)}. \tag{1.51}$$

Proof. Let $x_0 > 0$ and observe that (1.18) yields

$$h_t^{(-1)}(x_0, t) = \frac{1}{2} \frac{h_{xx}\left(h^{(-1)}(x_0, t)\right)}{h_x\left(h^{(-1)}(x_0, t), t\right)} = \frac{1}{2} \frac{\displaystyle\int_a^{\frac{1}{1-\gamma}} y^2 e^{yh^{(-1)}(x_0,t) - \frac{1}{2}y^2 t} \mu(dy)}{\displaystyle\int_a^{\frac{1}{1-\gamma}} y e^{yh^{(-1)}(x_0,t) - \frac{1}{2}y^2 t} \mu(dy)}$$

and thus inequality (1.51) holds, for all $t \geq 0$.

To show that $\lim_{t\uparrow\infty} h_t^{(-1)}(x_0, t)$ exists, it suffices to show that $h_t^{(-1)}(x_0, t)$ is decreasing in time. Indeed, direct calculations yield

$$h_{tt}^{(-1)}(x_0, t) = -\frac{\displaystyle\int_a^{\frac{1}{1-\gamma}} \left(y h_t^{(-1)}(x_0, t) - \frac{1}{2}y^2\right)^2 e^{yh^{(-1)}(x_0,t) - \frac{1}{2}y^2 t} \mu(dy)}{\displaystyle\int_a^{\frac{1}{1-\gamma}} y e^{yh^{(-1)}(x_0,t) - \frac{1}{2}y^2 t} \mu(dy)} < 0. \tag{1.52}$$

Alternatively, differentiating $h\left(h^{(-1)}(x_0, t), t\right) = x_0$ twice yields, setting $z = h^{(-1)}(x_0, t)$,

$$h_{tt}^{(-1)}(x_0, t) h_x(z, t) + \left(h_t^{(-1)}(x_0, t)\right)^2 h_{xx}(z, t) + 2h_t^{(-1)}(x_0, t)\, h_{xt}(z, t) + h_{tt}(z, t) = 0.$$

We have that both h_x, $h_{xx} > 0$, as it follows directly from (1.19) and differentiation. Furthermore, the above quadratic in $h_t^{(-1)}(x, t)$ remains positive, which would then yield that $h_{tt}^{(-1)}(x_0,t)<0$. Indeed,

$$h_{xt}^2\,(z,t)-h_{xx}\,(z,t)\,h_{tt}\,(z,t)=h_{xxx}^2\,(z,t)-h_{xx}\,(z,t)\,h_{xxxx}\,(z,t)<0,$$

as it follows from (1.20). $\square$

We are now ready to present one of the main findings herein, which yields the limit as $t\uparrow\infty$ of the ratio $\frac{1}{t}h^{(-1)}(x_0,t)$. We show that it converges to half of the lower-end of the measure's support. Some related weaker results can be found in [63].

Proposition 1.4.1. *Let $h^{(-1)}:\mathbb{D}_+\to\mathbb{R}$ be the spatial inverse of the function h (cf. (1.19)) and let a,b be the left and right end of the support, respectively, with $a=0^+$ or $a>0$, and $b<\infty$. Then, for each $x_0>0$, the following assertions hold.*

i) It holds that

$$\lim_{t\uparrow\infty}\frac{h^{(-1)}(x_0,t)}{t}=\frac{a}{2}. \tag{1.53}$$

ii) Let

$$\Delta\,(x_0,t):=\frac{h^{(-1)}(x_0,t)}{t}-\frac{a}{2}. \tag{1.54}$$

If $a>0$, then

$$|\Delta\,(x_0,t)|\leq\frac{1}{at}\ln\left(\frac{\mu\left([a,\frac{1}{1-\gamma}]\right)}{x_0}\right),\quad \textit{if}\ \ \Delta\,(x_0,t)<0, \tag{1.55}$$

and

$$x_0\geq\mu\left([a,a+\Delta\,(x_0,t)]\right)e^{\frac{1}{2}ta\Delta(x_0,t)},\quad \textit{if}\ \ \Delta\,(x_0,t)>0. \tag{1.56}$$

If $a = 0^+$, then $\Delta(x_0,t) > 0$, and, moreover, for each $\theta \in (0,1)$,

$$x_0 \geq \mu\left(\left[\Delta(x_0,t), (1+\theta)\Delta(x_0,t)\right]\right) e^{\frac{1}{2}t\left(1-\theta^2\right)\Delta^2(x_0,t)}. \tag{1.57}$$

Proof. i). Let $x_0 > 0$ fixed. Recall that $h_t^{(-1)}(x_0,t) > 0$ (cf. (1.51)) and, thus, $\lim_{t\uparrow\infty} h^{(-1)}(x_0,t)$ exists. Moreover, rewriting (1.50) as

$$x_0 = \int_a^{\frac{1}{1-\gamma}} e^{ty\left(\frac{h^{(-1)}(x_0,t)}{t} - \frac{1}{2}y\right)} \mu(dy), \tag{1.58}$$

we see that $\lim_{t\uparrow\infty} h^{(-1)}(x_0,t) = \infty$, otherwise, sending $t\uparrow\infty$ we get a contradiction. In turn, from Lemma 7 and L' Hospital's rule, we deduce that

$$A(x_0) := \lim_{t\uparrow\infty} \frac{h^{(-1)}(x_0,t)}{t} = \lim_{t\uparrow\infty} h_t^{(-1)}(x_0,t), \tag{1.59}$$

and thus

$$\frac{a}{2} \leq A(x_0) \leq \frac{1}{2(1-\gamma)}. \tag{1.60}$$

Next, we claim that $A(x_0) < \frac{1}{2(1-\gamma)}$.

Let $a > 0$. If $a = \frac{1}{1-\gamma}$, then $a = b$ and $h^{(-1)}(x_0,t) = \ln x_0^{1-\gamma} + \frac{1}{2}\frac{1}{1-\gamma}t$, and the result follows directly.

Let $0 < a < \frac{1}{1-\gamma}$. Assume that there exists x_0 such that $A(x_0) = \frac{1}{2(1-\gamma)}$. Then, for $\varepsilon > 0$, there exists $t_0(x_0,\varepsilon)$ such that, for $t \geq t_0$,

$$-\varepsilon \leq \frac{h^{(-1)}(x_0,t)}{t} - \frac{1}{2(1-\gamma)} \leq \varepsilon.$$

In turn, for $\delta > 0$ small enough, the above and (1.50) yield

$$x_0 \geq \int_a^{\left(\frac{1}{1-\gamma}-2\varepsilon-\delta\right)^-} e^{ty\left(\frac{1}{2(1-\gamma)}-\varepsilon-\frac{1}{2}y\right)}\mu(dy) + \int_{\frac{1}{1-\gamma}-2\varepsilon-\delta}^{\frac{1}{1-\gamma}} e^{ty\left(\frac{1}{2(1-\gamma)}-\varepsilon-\frac{1}{2}y\right)}\mu(dy),$$

which yields a contradiction as $t \uparrow \infty$, because the first integral would converge to ∞.

Next, assume that there exists $x_0 > 0$ such that

$$\frac{a}{2} < A(x_0) < \frac{1}{2(1-\gamma)}. \tag{1.61}$$

Then, for $\varepsilon, \delta > 0$ small enough we have

$$a < 2(A(x_0) - \varepsilon) - \delta < 2(A(x_0) - \varepsilon) < \frac{1}{1-\gamma}. \tag{1.62}$$

From (1.50), we then deduce that, for $t \geq t_0(x_0, \varepsilon)$, $x_0 \geq \int_a^{\frac{1}{1-\gamma}} e^{t(y(A(x_0)-\varepsilon)-\frac{1}{2}y^2)}\mu(dy)$. If $\mu(\{a\}) \neq 0$, then $x_0 \geq e^{\frac{ta}{2}(2(A(x_0)-\varepsilon)-a)}\mu(\{a\})$, and sending $t \uparrow \infty$ yields a contradiction. If $\mu(\{a\}) = 0$, then

$$x_0 \geq \int_a^{\frac{1}{1-\gamma}} e^{t(y(A(x_0)-\varepsilon)-\frac{1}{2}y^2)}\mu(dy) \geq \int_a^{2(A(x_0)-\varepsilon)-\delta} e^{t(y(A(x_0)-\varepsilon)-\frac{1}{2}y^2)}\mu(dy). \tag{1.63}$$

Consider the quadratic $B(y) = y(A(x_0) - \varepsilon) - \frac{1}{2}y^2$. We have

$$B(y_1) = B(y_2) = 0, \text{ for } \; y_1 = 0 \; \text{ and } \; y_2 = 2(A(x_0) - \varepsilon),$$

$B(y) > 0$, for $0 < y < 2(A(x_0) - \varepsilon)$, and $B(y)$ achieves a maximum at $y^* = A(x_0) - \varepsilon$.

Next, we look at its minimum, $y_* = \min_{a \leq y \leq 2(A(x)-\varepsilon)-\delta} \Delta(y)$, and claim that

$$y_* = 2(A(x_0) - \varepsilon) - \delta. \tag{1.64}$$

Indeed, if $0 < a \leq y^*$, then choosing $\delta < a$, direct calculations yield $\Delta(a) > \Delta(y_*)$. If $y^* < a$, then (1.62) yields $a < y_* < y_2$, and, thus, the minimum also occurs at y_*.

Clearly, because $y_1 < y_* < y_2$, we have $B(y_*) = \frac{1}{2}\delta\left(2(A(x_0)-\varepsilon)-\delta\right) > 0$. Therefore, for $t \geq t_0(x_0,\varepsilon)$,

$$x_0 \geq \int_a^{2(A(x_0)-\varepsilon)-\delta} e^{tB(y_*)}\mu(dy). \tag{1.65}$$

As $t \uparrow \infty$, the right hand side of (1.65) converges to ∞, unless it holds that $\mu\left([a, 2(A(x_0)-\varepsilon)-\delta]\right) = 0$. Sending $\delta \downarrow 0$ and $\varepsilon \downarrow 0$, we then have

$$\mu([a, 2A(x_0)]) = 0,$$

which, however, contradicts (1.61). Therefore, it must be that that, for all $x > 0$, $A(x_0) \leq \frac{a}{2}$, and we easily conclude.

If $a = 0^+$, similar arguments yield that for every $\theta \in (0, A(x_0)]$, we have that $\mu([\theta, 2A(x_0)]) = 0$. Sending $\theta \downarrow 0$ yields $\mu\left(0, 2A(x_0)\right] = 0$, which contradicts (1.61).

ii). Let $a > 0$.

If $\Delta(x_0,t) < 0$, from (1.50) we have

$$\begin{aligned} x_0 &= \int_a^{\frac{1}{1-\gamma}} e^{ty\left(\Delta(x_0,t)+\frac{1}{2}(a-y)\right)}\mu(dy) \\ &\leq e^{ta\Delta(x_0,t)}\int_a^{\frac{1}{1-\gamma}} e^{\frac{1}{2}ty(a-y)}\mu(dy) \leq e^{ta\Delta(x_0,t)}\mu\left(\left[a, \frac{1}{1-\gamma}\right]\right), \end{aligned}$$

and (1.55) follows.

If $\Delta(x_0,t) > 0$, then (1.53) yields that, for ε small enough and $t \geq t_0(x_0,\varepsilon)$, $0 < \frac{h^{(-1)}(x_0,t)}{t} - \frac{a}{2} < \varepsilon$. Choosing ε such that $\varepsilon < \frac{1}{2(1-\gamma)} - \frac{a}{2}$ yields $0 < \frac{h^{(-1)}(x_0,t)}{t} - \frac{a}{2} < \frac{1}{2(1-\gamma)} - \frac{a}{2}$, and using that $a < \frac{1}{1-\gamma}$, gives

$$\frac{a}{2} + \frac{h^{(-1)}(x_0,t)}{t} \leq \frac{1}{1-\gamma}.$$

From (1.28) we then deduce that

$$x_0 \geq \int_a^{\frac{a}{2}+\frac{h(-1)(x_0,t)}{t}} e^{ty\left(\frac{h^{(-1)}(x_0,t)}{t}-\frac{y}{2}\right)}\mu\left(dy\right).$$

The quadratic $H\left(y\right) := y\left(\frac{h^{(-1)}(x_0,t)}{t}-\frac{y}{2}\right)$ in the above integrand becomes zero at $y_1 = 0$ and $y_3 = 2\frac{h^{(-1)}(x_0,t)}{t} > a$ and, therefore, its minimum occurs at one of the end points a or $\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}$. Note that $a < \frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t} < y_3$.

If it occurs at a, then $H\left(a\right) = a\Delta\left(x_0,t\right)$, while if it occurs at $\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}$, then $H\left(\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}\right) = \frac{1}{2}\left(\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}\right)\Delta\left(x_0,t\right) > \frac{1}{2}a\Delta\left(x_0,t\right).$

Combining the above gives

$$x_0 \geq \int_a^{\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}} e^{\frac{1}{2}ta\Delta(x_0,t)}\mu\left(dy\right) = \mu\left(\left[a, a+\Delta\left(x_0,t\right)\right]\right)e^{\frac{1}{2}ta\Delta(x_0,t)}.$$

Finally, let $a = 0^+$. Then, $\Delta\left(x_0,t\right) = \frac{h^{(-1)}(x_0,t)}{t}$.

Recall that $\lim_{t\uparrow\infty} h^{(-1)}\left(x_0,t\right) = \infty$, and thus $\frac{h^{(-1)}(x_0,t)}{t} > 0$, for t large. For $\varepsilon \in \left(\frac{h^{(-1)}(x_0,t)}{t}, 2\frac{h^{(-1)}(x_0,t)}{t}\right)$ we then have

$$x_0 \geq \int_{\frac{h^{(-1)}(x_0,t)}{t}}^{\varepsilon} e^{ty\left(\frac{h^{(-1)}(x_0,t)}{t}-\frac{y}{2}\right)}\mu\left(dy\right) \geq \int_{\frac{h^{(-1)}(x_0,t)}{t}}^{\varepsilon} e^{t\varepsilon\left(\frac{h^{(-1)}(x_0,t)}{t}-\frac{\varepsilon}{2}\right)}\mu\left(dy\right).$$

Setting $\varepsilon = \left(1+\theta\right)\frac{h^{(-1)}(x_0,t)}{t}$, (1.57) follows. □

We are now ready to prove one of the main results herein.

Theorem 1.4.1. *Let a be the left end of the support of the measure μ. Then, for each $x_0 > 0$,*

$$\lim_{t\uparrow\infty}\frac{r\left(x_0,t\right)}{x_0} = a. \tag{1.66}$$

Furthermore, there exists a function $G(x_0,t)$ *given by*

$$G(x_0,t) := \begin{cases} \int_a^{\frac{1}{1-\gamma}} (y-a) e^{-ty\left(\frac{y-a}{2}\right)} \mu(dy), & \Delta(x_0,t) < 0 \\ 2\Delta(x_0,t)\, x_0 + \int_{a+2\Delta(x_0,t)}^{\frac{1}{1-\gamma}} (y-a) e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)} \mu(dy), & \Delta(x_0,t) > 0, \end{cases}$$

satisfying with $\lim_{t\uparrow\infty} G(x_0,t) = 0$ *and, for* t *large enough,*

$$0 \le r(x_0,t) - a x_0 \le G(x_0,t). \tag{1.67}$$

Proof. We present two alternative convergence proofs. The first yields (1.66) while the second gives the rate of convergence $G(x_0,t)$.

To this end, differentiating (1.17) gives

$$u_{xt}(x_0,t) = \left(\frac{1}{2} - h_t^{(-1)}(x_0,t)\right) u_x(x_0,t). \tag{1.68}$$

Moreover, (1.14) and (1.16) imply that $u_t(x_0,t) = -\frac{1}{2} u_x(x_0,t) r(x_0,t)$ and, in turn,

$$u_{tx}(x_0,t) = -\frac{1}{2} u_{xx}(x_0,t)\, r(x_0,t) - \frac{1}{2} u_x(x_0,t)\, r_x(x_0,t). \tag{1.69}$$

Combining the above we deduce

$$\frac{1}{2} r_x(x_0,t) = h_t^{(-1)}(x_0,t), \tag{1.70}$$

and from Proposition 8 and (1.59)

$$\lim_{t\uparrow\infty} r_x(x_0,t) = \lim_{t\uparrow\infty} 2h_t^{(-1)}(x_0,t) = a. \tag{1.71}$$

On the other hand,

$$\lim_{c\downarrow 0^+} \int_c^{x_0} r_x(\rho,t) d\rho = r(x_0,t) - \lim_{c\downarrow 0^+} r(c,t).$$

Using the fact that, for all $t \geq 0$, $\lim_{x \downarrow 0^+} r(x,t) = 0$ (see [58]), we get that, for $x_0 > 0$,

$$r(x_0, t) = \int_a^{x_0} r_x(\rho, t) d\rho. \tag{1.72}$$

Finally, we deduce from (1.70) and (1.52) that $r_{xt}(x_0, t) < 0$, and thus, for $x_0 > 0$, we have for $y \in (0, x_0]$, $r_x(y,t) \leq r_x(x_0, 0)$. However, for all $x_0 > 0$, $r_x(x_0, 0) < \infty$. This follows directly from (2.13),(1.19) and the full range of $h(x, 0)$, since

$$r_x(h(z,0), 0) = \frac{h_{zz}(z,0)}{h_z(z,0)} = \frac{\int_a^{\frac{1}{1-\gamma}} y^2 e^{yz - \frac{1}{2}t^2 y} \mu(dy)}{\int_a^{\frac{1}{1-\gamma}} y e^{yz - \frac{1}{2}t^2 y} \mu(dy)} \leq \frac{1}{1-\gamma}.$$

Using the dominated convergence theorem and passing to the limit as $t \uparrow \infty$ in (1.70), we deduce (1.66).

Next, we give the second convergence proof, which also yields the rate of convergence. First note that

$$0 \leq r(x_0, t) - a x_0. \tag{1.73}$$

This follows directly from (2.13), (1.19) and (1.50), for

$$r(x_0, t) = \int_a^{\frac{1}{1-\gamma}} y e^{t\left(y \frac{h^{(-1)}(x_0,t)}{t} - \frac{1}{2} y^2\right)} \mu(dy) \geq a \int_a^{\frac{1}{1-\gamma}} e^{t\left(y \frac{h^{(-1)}(x_0,t)}{t} - \frac{1}{2} y^2\right)} \mu(dy).$$

Furthermore, from (2.13), (1.19), (1.50) and (1.54), we have

$$r(x_0, t) - a x_0 = \int_a^{\frac{1}{1-\gamma}} (y - a) e^{ty\left(\frac{2\Delta(x_0,t) + a - y}{2}\right)} \mu(dy). \tag{1.74}$$

If $\Delta(x_0, t) < 0$ (which occurs only if $a > 0$, as shown in the previous proof), then the above yields

$$r(x_0, t) - a x_0 \leq \int_a^{\frac{1}{1-\gamma}} (y - a) e^{-ty\left(\frac{y-a}{2}\right)} \mu(dy),$$

and (1.67) follows directly with $G(t) := \int_a^{\frac{1}{1-\gamma}} (y-a)e^{-ty(\frac{y-a}{2})}\mu(dy)$.

Let $\Delta(x_0,t) > 0$ and $a > 0$ or $a = 0^+$. If $a = \frac{1}{1-\gamma}$, then the result follows trivially.

For $a < \frac{1}{1-\gamma}$, observe that for t large enough, $0 < a + 2\Delta(x_0,t) < \frac{1}{1-\gamma}$, and thus representation (1.74) gives

$$r(x_0,t) - ax_0 = \int_a^{(a+2\Delta(x_0,t))^-} (y-a)e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy)$$

$$+ \int_{a+2\Delta(x_0,t)}^{\frac{1}{1-\gamma}} (y-a)e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy).$$

Let $C_1(x_0,t) := \int_a^{(a+2\Delta(x_0,t))^-} (y-a)e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy)$, and observe that

$$C_1(x_0,t) \le 2\Delta(x_0,t) \int_a^{(a+2\Delta(x_0,t))^-} e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy) \le 2\Delta(x_0,t)\, x_0,$$

where we used (1.50). Thus

$$\lim_{t\uparrow\infty} C_1(x_0,t) = 0. \tag{1.75}$$

Let also $C_2(x_0,t) := \int_{a+2\Delta(x_0,t)}^{\frac{1}{1-\gamma}} (y-a)e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy)$ and $F(y,t,x_0) := (y-a)\, e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}$, $y \in \left[a + 2\Delta(x_0,t), \frac{1}{1-\gamma}\right]$. Then, $F(a+2\Delta(x_0,t), t, x_0) = 2\Delta(x_0,t)$, and thus $\lim_{t\uparrow\infty} F(a+2\Delta(x_0,t), t, x_0) = 0$. Furthermore, for each $y \in \left(a + 2\Delta(x_0,t), \frac{1}{1-\gamma}\right]$, we also have $\lim_{t\uparrow\infty} F(y,t,x_0) = 0$. In turn, the dominated convergence theorem gives

$$\lim_{t\uparrow\infty} C_2(x_0,t) = 0. \tag{1.76}$$

Setting $G(x_0,t) := C_1(x_0,t) + C_2(x_0,t)$, and using (3.4) and (1.76), we obtain (1.67). $\square$

1.5 Spatial and temporal limits for the relative prudence function

We now revert our attention to the relative prudence function $p(x,t)$ defined, for $(x,t) \in \mathbb{D}_+$, as

$$p(x,t) = -\frac{x u_{xxx}(x,t)}{u_{xx}(x,t)}, \tag{1.77}$$

with u solving (1.14).

Proposition 1.5.1. *For $(x,t) \in \mathbb{D}_+$, we have that $p(x,t) > 0$. Moreover, the following spatial and temporal limits hold.*

i) If Assumption 1 holds, then, for each $t_0 \geq 0$,

$$\lim_{x\uparrow\infty} p(x,t_0) = 2 - \gamma. \tag{1.78}$$

ii) If Assumption 2 holds, then, for each $x_0 > 0$,

$$\lim_{t\uparrow\infty} p(x_0,t) = \begin{cases} 1 + \frac{1}{a}, & \text{if } \ a > 0 \\ \\ \infty, & \text{if } a = 0^+. \end{cases} \tag{1.79}$$

Proof. Using (1.77) and (1.16), we deduce that, for each $t_0 \geq 0$,

$$p(x,t_0) = \frac{x}{r(x,t_0)}\left(1 + r_x(x,t_0)\right),$$

and the fact that $p(x,t_0) > 0$ and (1.78) follow directly from (1.23) and (1.37), respectively.

From (1.77) and equation (1.14) we also obtain that, for each $x_0 > 0$,

$$\frac{u_{xt}(x_0,t)}{u_x(x_0,t)} = 1 - \frac{1}{2}\frac{r(x_0,t)}{x_0}p(x_0,t) = \frac{1}{2} - h_t^{(-1)}(x_0,t). \tag{1.80}$$

Using that $\lim_{t\uparrow\infty} h_t^{(-1)}(x_0,t) = \frac{a}{2}$ we easily conclude. $\square$

1.6 Examples

We present two representative examples in which the measure is, respectively, a sum of Dirac functions and the Lebesgue measure. The first example generalizes the results of the example in subsection 2.1, while the second demonstrates that the spatial turnpike property fails if there is no mass at the right end of the measure's support.

1.6.1 Finite sum of Dirac functions

We assume that

$$\mu = \sum_{n=1}^{N} \delta_{y_n}, \quad \text{with} \quad 0 < y_1 < \cdots < y_N = \frac{1}{1-\gamma}.$$

Then, $h(z,0) = \sum_{n=1}^{N} e^{y_n z}$ and, thus, $\lim_{z\uparrow\infty} h(z,0)\, e^{-zy_N} = 1$. In turn, (1.34) yields

$$\lim_{x\uparrow\infty} \frac{u_x(x,0)}{x^{\gamma-1}} = 1,$$

which verifies the results of Lemma 2. We also have, for $(z,t) \in \mathbb{D}$,

$$h(z,t) = \sum_{n=1}^{N} \exp\left(y_n z - \frac{1}{2} y_n^2 t\right).$$

(cf. (1.19)), and, therefore, for $x > 0$,

$$x = \sum_{n=1}^{N} \exp\left(y_n t \left(\frac{h^{(-1)}(x,t)}{t} - \frac{1}{2} y_n\right)\right). \tag{1.81}$$

Furthermore,

$$h^{(-1)}(x,t) - \frac{1}{2} y_1 t \le \frac{1}{y_1} \ln x. \tag{1.82}$$

1.6.1.1 Temporal asymptotic expansion of $h^{(-1)}(x_0, t)$ for large t

We claim that, for each $x_0 > 0$, as $t \uparrow \infty$,

$$h^{(-1)}(x_0, t) = \frac{1}{2} y_1 t + \frac{1}{y_1} \ln x_0 + o(1). \tag{1.83}$$

Indeed, using the limit (1.53), we have

$$\lim_{t \uparrow \infty} \left(\frac{h^{(-1)}(x_0, t)}{t} - \frac{1}{2} y_n \right) \begin{cases} < 0, & 1 < n \leq N \\ = 0, & n = 1. \end{cases}$$

Therefore, as $t \uparrow \infty$, all the terms in (1.81) vanish except for the first one, and thus,

$$x_0 = \lim_{t \uparrow \infty} \exp \left(y_1 h^{(-1)}(x_0, t) - \frac{1}{2} y_1^2 t \right). \tag{1.84}$$

Taking logarithm and rearranging terms yields (1.83).

1.6.1.2 Spatial asymptotic expansion of $h^{(-1)}(x, t_0)$ for large x

We claim that, for each $t_0 \geq 0$,

$$h^{(-1)}(x, t_0) = (1 - \gamma) \ln x + \frac{1}{2(1 - \gamma)} t_0 + o(1). \tag{1.85}$$

To obtain this, we first establish that

$$\lim_{x \uparrow \infty} \frac{h^{(-1)}(x, t_0)}{\ln x} = (1 - \gamma). \tag{1.86}$$

Indeed, fix $t_0 \geq 0$, let $\delta \in (0, \frac{1}{1-\gamma})$ and assume that

$$\liminf_{x \uparrow \infty} \frac{h^{(-1)}(x, t_0)}{\ln x} < \frac{1}{\frac{1}{1-\gamma} + \delta}.$$

Then, using (1.81) and that $h^{(-1)}(x,t_0) > 0$, for large x, we have

$$1 = \frac{1}{x}\sum_{n=1}^{N} \exp\left(y_n \ln x \left(\frac{h^{(-1)}(x,t_0)}{\ln x}\right) - \frac{1}{2}y_n^2 t_0\right)$$

$$\leq \frac{1}{x}\sum_{n=1}^{N} \exp\left(y_n \ln x \left(\frac{h^{(-1)}(x,t_0)}{\ln x}\right)\right) \leq N x^{\frac{1}{1-\gamma}\frac{h^{(-1)}(x,t_0)}{\ln x} - 1},$$

and using that $\frac{1}{1-\gamma}\frac{1}{\frac{1}{1-\gamma}+\delta} - 1 = -\frac{\delta(1-\gamma)}{1+\delta(1-\gamma)} < 0$, we get a contradiction as $x \uparrow \infty$.

Since δ is arbitrary, we deduce that

$$\liminf_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{\ln x} \geq (1-\gamma)\,. \tag{1.87}$$

Similarly, assume that for $\delta \in \left(0, \frac{1}{1-\gamma}\right)$,

$$\limsup_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{\ln x} > \frac{1}{\frac{1}{1-\gamma} - \delta}.$$

Then, (1.82) gives

$$1 > \frac{1}{x}\exp\left(\frac{1}{1-\gamma}\ln x \frac{h^{(-1)}(x,t_0)}{\ln x} - \frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0\right)$$

$$= x^{\frac{1}{1-\gamma}\frac{h^{(-1)}(x,t_0)}{\ln x} - 1} e^{-\frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0}$$

and using that $\frac{1}{1-\gamma}\frac{1}{\frac{1}{1-\gamma}-\delta} - 1 = \frac{\delta(1-\gamma)}{1-\delta(1-\gamma)} > 0$, we get a contradiction as $x \uparrow \infty$. Since δ is arbitrary, we deduce that

$$\limsup_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{\ln x} \leq (1-\gamma)\,, \tag{1.88}$$

and we easily conclude.

Next, we rewrite (1.81) as

$$1 = \sum_{n=1}^{N} \exp\left(y_n h^{(-1)}(x,t_0) - \frac{1}{2} y_n^2 t_0 - \ln x \right) \tag{1.89}$$

$$= \sum_{n=1}^{N} \exp\left(y_n \ln x \left(\frac{h^{(-1)}(x,t_0)}{\ln x} - \frac{1}{y_n} \right) - \frac{1}{2} y_n^2 t_0 \right).$$

Note that from the limit in (1.86) we have that

$$\lim_{x\uparrow\infty} \left(\frac{h^{(-1)}(x,t_0)}{\ln x} - \frac{1}{y_n} \right) = \begin{cases} < 0, & 1 \le n < N \\ = 0, & n = N. \end{cases}$$

Therefore, as $x \uparrow \infty$, the first $N-1$ terms in (1.89) vanish, and we deduce that

$$\lim_{x\uparrow\infty} \exp\left(\frac{1}{1-\gamma} h^{(-1)}(x,t_0) - \ln x - \frac{1}{2} \left(\frac{1}{1-\gamma} \right)^2 t_0 \right) = 1.$$

We then obtain (1.85) by taking the logarithm and rearranging the terms.

1.6.1.3 Spatial and temporal asymptotics of $r(x,t)$

From representation (2.13), we have for the risk tolerance function

$$r(x,t) = \sum_{n=1}^{N} y_n \exp\left(y_n h^{(-1)}(x,t) - \frac{1}{2} y_n^2 t \right). \tag{1.90}$$

Let $x_0 > 0$. Then, (1.82) gives

$$r(x_0,t) \le \sum_{n=1}^{N} y_n \exp\left(y_n (\frac{1}{2} y_1 t + \frac{1}{y_1} \ln x_0) - \frac{1}{2} y_n^2 t \right)$$

$$= y_1 x_0 + \sum_{n=2}^{N} y_n \exp\left(\frac{1}{2} y_n (y_1 - y_n) t \right) x_0^{\frac{y_n}{y_1}}.$$

Therefore, the temporal asymptotic expansion of $r(x_0, t)$ as $t \uparrow \infty$ is given by

$$r(x_0, t) = y_1 x_0 + O\left(e^{\frac{1}{2}y_2(y_1 - y_2)t}\right). \tag{1.91}$$

Next, let $t_0 \geq 0$. Then,

$$\lim_{x \uparrow \infty} r(x, t_0) = \lim_{x \uparrow \infty} \sum_{n=1}^{N} y_n \exp\left(y_n((1-\gamma)\ln x + \frac{1}{2(1-\gamma)}t_0) - \frac{1}{2}y_n^2 t_0\right),$$

and, thus, as $x \uparrow \infty$,

$$r(x, t_0) = \sum_{n=1}^{N} y_n \exp\left(\frac{1}{2}y_n t_0(\frac{1}{1-\gamma} - y_n)\right) x^{(1-\gamma)y_n} + o(1). \tag{1.92}$$

Therefore, for each $x_0 > 0$ and $t_0 \geq 0$, we have the temporal asymptotic expansion (1.91) yields

$$\lim_{t \uparrow \infty} \frac{r(x_0, t)}{x_0} = y_1 \quad \text{and} \quad \lim_{x \uparrow \infty} \frac{r(x, t_0)}{x} = y_N = \frac{1}{1-\gamma},$$

and these limits are consistent with the findings in Proposition 3 and Theorem 9, respectively.

1.6.2 Lebesgue measure

We consider a case of a measure with continuous support but without a mass at its right boundary. We derive the associated limits and also show that the spatial turnpike property fails.

- Lebesgue measure on $\left[a, \frac{1}{1-\gamma}\right]$, $a > 0$

Consider the functions $\varphi(z) := e^{-\frac{z^2}{2}}$ and $\Phi(z) := \int_{-\infty}^{z} \varphi(y)dy$, for $z \in \mathbb{R}$. Then, representations (1.19) and (1.50) yield, respectively,

$$h(z,t) = \int_a^{\frac{1}{1-\gamma}} e^{yz-\frac{1}{2}y^2 t} dy = \frac{e^{z^2/2t}}{\sqrt{t}} \int_{a\sqrt{t}-z/\sqrt{t}}^{\frac{1}{1-\gamma}\sqrt{t}-z/\sqrt{t}} \varphi(y)dy, \tag{1.93}$$

and

$$x = \int_a^{\frac{1}{1-\gamma}} e^{yt\left(\frac{h^{(-1)}(x,t)}{t}-\frac{1}{2}y\right)} dy = \frac{1}{\sqrt{t}} e^{\frac{h^{(-1)}(x,t)^2}{2t}} \int_{a\sqrt{t}-\frac{h^{(-1)}(x,t)}{\sqrt{t}}}^{\frac{1}{1-\gamma}\sqrt{t}-\frac{h^{(-1)}(x,t)}{\sqrt{t}}} \varphi(y)dy. \tag{1.94}$$

1.6.2.1 Temporal asymptotic expansion of $h^{(-1)}(x_0, t)$ for large t

We claim that for $x_0 > 0$, as $t \uparrow \infty$,

$$h^{(-1)}(x_0,t) = \frac{1}{2}at + \frac{1}{a}\left(\ln t + \ln x_0 + \ln \frac{a}{2}\right) + o(1). \tag{1.95}$$

To show this, we first establish that

$$x_0 = \lim_{t\uparrow\infty} \frac{e^{a(h^{(-1)}(x_0,t)-\frac{1}{2}at)}}{\frac{1}{2}at}. \tag{1.96}$$

Using (1.94) and that, for $z < 0$,

$$\Phi(z) \le -\frac{\varphi(z)}{z}, \tag{1.97}$$

we have, for t large enough,

$$x_0 \le \frac{1}{\sqrt{t}} \exp\left(\frac{h^{(-1)}(x_0,t)^2}{2t}\right) \Phi\left(-a\sqrt{t} + \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)$$

$$\le \frac{1}{\sqrt{t}} \frac{1}{a\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}} \exp\left(\frac{h^{(-1)}(x_0,t)^2}{2t}\right) \varphi\left(-a\sqrt{t} + \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)$$

$$= \frac{e^{a(h^{(-1)}(x_0,t)-\frac{1}{2}at)}}{at-h^{(-1)}(x_0,t)}.$$

In turn,

$$x_0 \leq \liminf_{t\uparrow\infty} \frac{e^{a(h^{(-1)}(x_0,t)-\frac{1}{2}at)}}{at-h^{(-1)}(x_0,t)}. \tag{1.98}$$

Next, we show that

$$x_0 \geq \limsup_{t\uparrow\infty} \frac{e^{a(h^{(-1)}(x_0,t)-\frac{1}{2}at)}}{at-h^{(-1)}(x_0,t)},$$

which with (1.98) will yield (1.96). To this end, we use that for any $b > a > 0$, the inequality

$$\Phi(b) - \Phi(a) \geq \frac{1}{b}\left(\varphi(a) - \varphi(b)\right)$$

holds. Let $1 < k < \frac{1}{a(1-\gamma)}$. From (1.94) and the above, we have, for t large enough, that

$$\begin{aligned}
x_0 &\geq \frac{1}{\sqrt{t}} e^{\frac{h^{(-1)}(x_0,t)^2}{2t}} \left(\Phi\left(ka\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right) - \Phi\left(a\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)\right) \\
&\geq \frac{1}{\sqrt{t}} \frac{1}{ka\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}} e^{\frac{h^{(-1)}(x_0,t)^2}{2t}} \\
&\quad \times \left(\varphi\left(a\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right) - \varphi\left(ka\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)\right) \\
&= \frac{1}{kat - h^{(-1)}(x_0,t)} \left(e^{a(h^{(-1)}(x_0,t)-\frac{1}{2}at)} - e^{ka(h^{(-1)}(x_0,t)-\frac{1}{2}kat)}\right).
\end{aligned}$$

From Proposition 8 and since $k > 1$, we have

$$\lim_{t\uparrow\infty} \frac{e^{ka(h^{(-1)}(x_0,t)-\frac{1}{2}kat)}}{kat-h^{(-1)}(x_0,t)} = \lim_{t\uparrow\infty} \frac{e^{ka^2t(\frac{h^{(-1)}(x_0,t)}{at}-\frac{k}{2})}}{at\left(k-\frac{h^{(-1)}(x_0,t)}{at}\right)} = 0.$$

Therefore,

$$x_0 \geq \limsup_{t\uparrow\infty} \frac{1}{kat - h^{(-1)}(x_0,t)} \left(e^{a(h^{(-1)}(x_0,t)-\frac{1}{2}at)} - e^{ka(h^{(-1)}(x_0,t)-\frac{1}{2}kat)} \right)$$

$$\geq \limsup_{t\uparrow\infty} \frac{e^{ka(h^{(-1)}(x_0,t)-\frac{1}{2}kat)}}{kat - h^{(-1)}(x_0,t)} - \lim_{t\uparrow\infty} \frac{e^{ka(h^{(-1)}(x_0,t)-\frac{1}{2}kat)}}{kat - h^{(-1)}(x_0,t)} = \limsup_{t\uparrow\infty} \frac{e^{a(h^{(-1)}(x_0,t)-\frac{1}{2}at)}}{kat - h^{(-1)}(x_0,t)},$$

and sending $k \downarrow 1$ we conclude.

Next, we utilize the Lambert-W function $W(x)$, defined as the inverse function of $F(x) = xe^x$, to derive the explicit asymptotic expansion of $h^{(-1)}(x_0,t)$ as $t \uparrow \infty$. Recalling the notation $\Delta(x_0,t) = h^{(-1)}(x_0,t) - \frac{1}{2}at$, we deduce from (1.96) that there exists $\varepsilon(t)$ with $\lim_{t\uparrow\infty} \varepsilon(t) = 0$, such that

$$\frac{e^{a\Delta(x_0,t)}}{\frac{1}{2}at - \Delta(x_0,t)} = x_0(1+\varepsilon(t)).$$

Rewriting it yields

$$a\left(\frac{1}{2}at - \Delta(x_0,t)\right) e^{a(\frac{1}{2}at - \Delta(x_0,t))} = \frac{a}{x_0(1+\varepsilon(t))} e^{\frac{1}{2}a^2 t},$$

Using that the left hand side is of the form $F(a(\frac{1}{2}at - \Delta(x_0,t)))$, we obtain

$$a(\frac{1}{2}at - \Delta(x_0,t)) = W\left(\frac{a}{x_0(1+\varepsilon(t))} e^{\frac{1}{2}a^2 t}\right),$$

and, in turn,

$$\Delta(x_0,t) = \frac{1}{2}at - \frac{1}{a} W\left(\frac{a}{x_0(1+\varepsilon(t))} e^{\frac{1}{2}a^2 t}\right).$$

It is established in [18] that the asymptotic expansion of $W(x)$, for large x, is given by

$$W(x) = \ln x - \ln(\ln x) + o(1).$$

Therefore,

$$\Delta(x_0,t) = \frac{1}{2}at - \frac{1}{a}\ln\left(\frac{a}{x_0(1+\varepsilon(t))}e^{\frac{1}{2}a^2t}\right) + \frac{1}{a}\ln\ln\left(\frac{a}{x_0(1+\varepsilon(t))}e^{\frac{1}{2}a^2t}\right) + o(1)$$
$$= \frac{1}{a}\left(\ln\frac{x_0}{a} + \ln(1+\varepsilon(t)) + \ln\left(\frac{1}{2}a^2t + \ln\frac{a}{x_0(1+\varepsilon(t))}\right)\right) + o(1).$$

Using that as $t\uparrow\infty$, $\ln(1+\varepsilon(t)) = o(1)$ and that

$$\ln\left(\frac{1}{2}a^2t + \ln\frac{a}{x_0(1+\varepsilon(t))}\right) = \ln\left(\frac{1}{2}a^2t\right) + o(1),$$

assertion (1.95) follows.

1.6.2.2 Spatial asymptotic expansion of $h^{(-1)}(x,t_0)$ for large x

Let $t_0 \geq 0$. We show that, as $x\uparrow\infty$,

$$h^{(-1)}(x,t_0) = \frac{1}{2(1-\gamma)}t_0 + (1-\gamma)\left(\ln x + \ln\ln x - \ln\frac{1}{1-\gamma}\right) + o(1). \quad (1.99)$$

We first establish that

$$\lim_{x\uparrow\infty}\frac{h^{(-1)}(x,t_0)}{\ln x} = (1-\gamma)\,. \quad (1.100)$$

Indeed, let $f(z,t) := \frac{1}{z}e^{\frac{1}{1-\gamma}z - \frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t}$. Then,

$$\lim_{z\uparrow\infty}\frac{h(z,t_0)}{f(z,t_0)} = \lim_{z\uparrow\infty}\int_a^{\frac{1}{1-\gamma}} ze^{z(y-\frac{1}{1-\gamma}) - \frac{1}{2}(y^2 - \left(\frac{1}{1-\gamma}\right)^2)t_0}dy$$

$$= \lim_{z\uparrow\infty}\left(\int_a^{\frac{1}{1-\gamma}}(z-yt_0)e^{z(y-\frac{1}{1-\gamma}) - \frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy + \int_a^{\frac{1}{1-\gamma}} yt_0e^{z(y-\frac{1}{1-\gamma}) - \frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy\right)$$

$$= \lim_{z\uparrow\infty}\left(1 - e^{(a-\frac{1}{1-\gamma})z - \frac{1}{2}(a^2-\left(\frac{1}{1-\gamma}\right)^2)t_0} + \int_a^{\frac{1}{1-\gamma}} yt_0e^{z(y-\frac{1}{1-\gamma}) - \frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy\right) = 1,$$

where we used that $a < \frac{1}{1-\gamma}$ and, for the third term, the monotone convergence theorem. Therefore, for each $t_0 \geq 0$,

$$\lim_{x\uparrow\infty} \frac{h(x,t_0)}{f\left(x,t_0\right)} = 1. \tag{1.101}$$

We now use a result on the inverses of asymptotic functions (see [23]) to prove the limit in (1.100) by verifying the necessary assumptions for this result to hold. To this end, consider the function $g(z) := (1-\gamma)\ln z$, and notice that

$$g(f(z,t_0)) = -\left(1-\gamma\right)\ln z + z - \frac{1}{2\left(1-\gamma\right)}t_0 \sim z, \quad \text{as} \quad z\uparrow\infty.$$

Thus, $\lim_{z\uparrow\infty} z^{-1}g(f(z,t_0)) = 1$. Since, on the other hand, $\lim_{z\uparrow\infty} f(z,t_0) = \infty$, we deduce that $f^{(-1)}(x,t_0) \sim g(x)$, as $x\uparrow\infty$. Moreover, $g(x)$ is strictly increasing and the ratio $\frac{g_x(x,t_0)}{g(x,t_0)} \sim \frac{1}{x\ln x} = O(\frac{1}{x})$, for sufficiently large x. It then follows from the aforementioned result that $g(x) \sim h^{(-1)}(x,t_0)$, as $x\uparrow\infty$, and (1.100) follows.

Next, we claim that, for each $t_0 \geq 0$,

$$\lim_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}\left(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0\right)}}{x\ln x} = 1-\gamma. \tag{1.102}$$

Indeed, for $t_0 = 0$, we have from (1.94) that

$$x = \int_a^{\frac{1}{1-\gamma}} e^{yh^{(-1)}(x,0)}dy = \frac{1}{h^{(-1)}(x,0)}\left(e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)} - e^{ah^{(-1)}(x,0)}\right), \tag{1.103}$$

and (1.100) yields that

$$\lim_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)}}{x\ln x}$$

$$= \lim_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)}}{e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)} - e^{ah^{(-1)}(x,0)}} \frac{h^{(-1)}(x,0)}{\ln x} = 1-\gamma.$$

For $t_0 > 0$, we deduce from (1.94) that

$$x = \frac{1}{\sqrt{t_0}} e^{-\frac{h^{(-1)}(x,t_0)^2}{2t_0}} \left(\Phi\left(\frac{1}{1-\gamma}\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}} \right) - \Phi\left(a\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}} \right) \right). \tag{1.104}$$

Then, using (1.97), we have, for large x,

$$1 \le \frac{1}{x\sqrt{t_0}} \exp\left(\frac{h^{(-1)}(x,t_0)^2}{2t_0} \right) \Phi\left(\frac{1}{1-\gamma}\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}} \right)$$

$$\le \frac{1}{x\sqrt{t_0}} e^{\frac{h^{(-1)}(x,t_0)^2}{2t_0}} \frac{1}{\frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}} - \frac{1}{1-\gamma}\sqrt{t_0}} \varphi\left(\frac{1}{1-\gamma}\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}} \right)$$

$$= \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0) - \frac{1}{1-\gamma}t_0)},$$

and, in turn,

$$1 \le \liminf_{x\uparrow\infty} \left(\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)} \frac{h^{(-1)}(x,t_0)}{h^{(-1)}(x,t_0) - \frac{1}{1-\gamma}t_0} \right)$$

$$= \liminf_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)} \lim_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{h^{(-1)}(x,t_0) - \frac{1}{1-\gamma}t_0}$$

$$= \liminf_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)}. \tag{1.105}$$

Similarly, we use that, for $a < b < 0$,

$$\Phi(b) - \Phi(a) \ge \frac{\varphi(a) - \varphi(b)}{a}, \tag{1.106}$$

and deduce from (1.104) that, for large x,

$$1 \geq \frac{1}{x\sqrt{t_0}} e^{\frac{h^{(-1)}(x,t_0)^2}{2t_0}} \frac{1}{a\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}}$$

$$\times \left(\varphi(a\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}) - \varphi(\frac{1}{1-\gamma}\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}) \right)$$

$$= \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0) - at_0)} - \frac{e^{a(h^{(-1)}(x,t_0) - \frac{1}{2}at_0)}}{x(h^{(-1)}(x,t_0) - at_0)}.$$

For the second term, we have

$$\lim_{x\uparrow\infty} \frac{e^{ah^{(-1)}(x,t_0)}}{x(h^{(-1)}(x,t_0) - at_0)} e^{-\frac{1}{2}at_0} = \lim_{x\uparrow\infty} \frac{e^{ah^{(-1)}(x,t_0) - \ln x}}{h^{(-1)}(x,t_0) - at_0} e^{-\frac{1}{2}at_0}$$

$$= \lim_{x\uparrow\infty} \exp\left(a \ln x \left(\frac{h^{(-1)}(x,t_0)}{\ln x} - \frac{1}{a} \right) \right) \frac{1}{h^{(-1)}(x,t_0) - at_0} = 0.$$

Therefore,

$$\limsup_{x\uparrow\infty} \left(\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0) - at_0)} - \frac{e^{a(h^{(-1)}(x,t_0) - \frac{1}{2}at_0)}}{x(h^{(-1)}(x,t_0) - at_0)} \right)$$

$$= \limsup_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0) - at_0)} - \lim_{x\uparrow\infty} \frac{e^{a(h^{(-1)}(x,t_0) - \frac{1}{2}at_0)}}{x(h^{(-1)}(x,t_0) - at_0)}$$

$$= \limsup_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0) - at_0)}$$

$$= \limsup_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)} \lim_{x\uparrow\infty} \frac{xh^{(-1)}(x,t_0)}{x(h^{(-1)}(x,t_0) - at_0)}$$

$$= \limsup_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)} \leq 1. \tag{1.107}$$

From (1.105) and (1.107), we then obtain

$$\limsup_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)} = 1,$$

which together with (1.105) gives (1.102). Taking the logarithm of both sides then yields

$$\lim_{x\uparrow\infty}\left(\frac{1}{1-\gamma}\left(h^{(-1)}(x,t_0)-\frac{1}{2(1-\gamma)}t_0\right)-\ln x-\ln\ln x\right)=\ln(1-\gamma),$$

and the spatial asymptotic expansion (1.99) follows.

1.6.2.3 Spatial asymptotics of $r(x,t_0)$ for large x

Let $t_0>0$. We show that as $x\uparrow\infty$, the spatial asymptotic expansion of $r(x,t_0)$ is given by

$$\begin{aligned}r(x,t_0)=&\frac{1-\gamma}{t_0}x\ln\ln x+\frac{1}{t_0}\left((1-\gamma)\,x\ln x\right)^{a(1-\gamma)}e^{\frac{1}{2}a(\frac{1}{1-\gamma}-a)t_0}\\&+\frac{1}{2(1-\gamma)}x-\frac{1-\gamma}{t_0}x\ln\frac{1}{1-\gamma}+o(1).\end{aligned}\tag{1.108}$$

Indeed, from (2.13) and (1.93), we have

$$\begin{aligned}r(x,t_0)=&\int_a^{\frac{1}{1-\gamma}}\frac{1}{t_0}(yt_0-h^{(-1)}(x,t_0))e^{yh^{(-1)}(x,t_0)-\frac{1}{2}y^2t_0}dy\\&+\frac{h^{(-1)}(x,t_0)}{t_0}\int_a^{\frac{1}{1-\gamma}}e^{yh^{(-1)}(x,t_0)-\frac{1}{2}y^2t_0}dy\\=&\frac{1}{t_0}\left(e^{a(h^{(-1)}(x,t_0)-\frac{1}{2}at_0)}-e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2(1-\gamma)}t_0)}\right)+\frac{h^{(-1)}(x,t_0)}{t_0}x,\end{aligned}$$

where we used (1.50) for the last term. Then, (1.108) follows using (1.99).

For $t_0=0$, we have from (1.103) that

$$r(x,0)=\frac{1}{1-\gamma}x-\frac{(\frac{1}{1-\gamma}-a)e^{ah^{(-1)}(x,0)}-x}{h^{(-1)}(x,0)},$$

and, for large x,

$$r(x,0) = \frac{1}{1-\gamma}x\left(1-\frac{1}{\ln x}\right)+o(1). \tag{1.109}$$

From (1.108) and (1.109), we then obtain that for $t_0>0$ and $t_0=0$, we have respectively,

$$r(x,t_0) \sim \frac{1-\gamma}{t_0}x\ln\ln x \qquad \text{and} \qquad r(x,0) \;\sim\; \frac{1}{1-\gamma}x.$$

Therefore, the risk tolerance function does *not* have the spatial turnpike property (1.37). Recall that the underlying measure *lacks* a Dirac mass on the right boundary of the measure μ, which is a necessary condition for the results in Proposition 3 to hold.

- The case $a=0^+$

We conclude with the case that μ is the Lebesgue measure on $(0,\frac{1}{1-\gamma}]$. For $t_0\geq 0$, we easily obtain the same spatial asymptotic expansions of $h^{(-1)}(x,t_0)$ as in (1.99) and of $r(x,t_0)$ as in (1.108) and (1.109).

For the temporal expansion, we claim that as $t\uparrow\infty$,

$$\frac{h^{(-1)}(x_0,t)}{t} = \frac{\sqrt{\ln t+2\ln x_0-\ln 2\pi}}{\sqrt{t}}+o(\frac{1}{\sqrt{t}}). \tag{1.110}$$

To see this, first recall (cf. (1.50)) that

$$x_0 = \int_{0^+}^{\frac{1}{1-\gamma}} e^{y(h^{(-1)}(x_0,t)-\frac{1}{2}yt)}dy, \tag{1.111}$$

Taking the logarithm of both sides of (1.111) yields

$$2\ln x_0 = \left(\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)^2-\ln t \tag{1.112}$$

$$+2\ln\left(\Phi\left(\sqrt{t}(\frac{1}{1-\gamma}-\frac{h^{(-1)}(x_0,t)}{t})\right)-\Phi\left(-\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)\right).$$

Next, we claim that $l := \liminf_{t\uparrow\infty}\frac{h^{(-1)}(x_0,t)}{\sqrt{t}} = \infty$. Indeed, if $l < \infty$, then, as $t \uparrow \infty$, the above yields

$$2\ln x_0 = l^2 - \lim_{t\uparrow\infty}(\ln t) + 2\ln(1-\Phi(-l)) = -\infty,$$

which is a contradiction. Therefore, it must be that $l = \infty$, which combined with the fact that $\lim_{t\uparrow\infty}\frac{h^{(-1)}(x_0,t)}{t} = 0$, implies that as $t \uparrow \infty$, the third term on the right hand side of (1.112) converges to $2\ln\sqrt{2\pi}$. Thus, we obtain

$$2\ln x_0 = \lim_{t\uparrow\infty}\left(\left(\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)^2 - \ln t + 2\ln\sqrt{2\pi}\right),$$

from which we deduce that $h^{(-1)}(x_0,t) = \sqrt{t(\ln t + 2\ln x_0 - \ln 2\pi)} + o(\sqrt{t})$, and (1.110) follows.

1.7 Extensions

We have analyzed the spatial and temporal asymptotic behavior of the risk tolerance function $r(x,t)$. We recall that the optimal portfolio process $\pi_t^{*,x}$ is given in the feedback form $\pi_t^{*,x} = \sigma_t^+\lambda_t r\left(X_t^{*,x},t\right)$, with $X_t^{*,x}$ being the wealth generated by it. Furthermore, it was shown in [58] that $X_t^{*,x}$ and $\pi_t^{*,x}$ are given in the closed form

$$X_t^{*,x} = h\left(h^{(-1)}(x,0)+A_t+M_t,A_t\right),\ \pi_t^{*,x} = \sigma_t^+\lambda_t h_x\left(h^{(-1)}(x,0)+A_t+M_t,A_t\right).$$

It is then natural to investigate the long-term limits $\lim_{t\uparrow\infty}X_t^{*,x}$, $\lim_{t\uparrow\infty}\pi_t^{*,x}$ under asymptotic assumptions on the initial datum and the results obtained herein. The asymptotic behavior of these processes has been investigated in [33] for the classical setting.

In a different direction, an interesting problem is how to construct investment policies which yield a targeted long-term wealth distribution. In a static model, this question was analyzed in [65] and in the log-normal, classical and forward cases, in [54]. However, in these settings, there is a strong model commitment, which is a nonrealistic assumption for long-term portfolio management.

In the forward setting we have analyzed herein, the model is dynamically updated. Furthermore, the distribution of the optimal wealth is given explicitly, using the above formula, by

$$\begin{aligned}\mathbb{P}\left(X_t^{*,x} \leq y\right) &= \mathbb{P}\left(h^{(-1)}\left(x,0\right) + A_t + M_t \leq h^{(-1)}\left(y, A_t\right)\right) \\ &= \mathbb{P}\left(\frac{h^{(-1)}\left(x,0\right)}{\langle M\rangle_t} + 1 + \frac{M_t}{\langle M\rangle_t} \leq \frac{h^{(-1)}\left(y,A_t\right)}{\langle M\rangle_t}\right),\end{aligned}$$

where we used that $A_t = \langle M\rangle_t$ (cf. (1.15)). Therefore, one expects that the limit (1.53) as well as results on strong law of large numbers for martingales can be used to study the long-term distribution of the optimal processes. Such questions are currently investigated by the authors in [29] and others.

Chapter 2

On the optimal wealth and portfolio weights processes under time-monotone forward performance criteria in an Itô-diffusion market

2.1 Introduction

This paper is a contribution to the forward portfolio theory where the optimal portfolio choice is determined in an Itô-diffusion market under the forward investment performance criterion. This criterion was first introduced by Musiela and Zariphopoulou in [55] and subsequently developed in [44, 56, 57, 59, 58, 60, 61, 73]. Under the forward performance criterion, an investment strategy is considered optimal if it generates a wealth process whose average performance is maintained over time. Such a criterion also offers flexibility for performance measurement and risk management under model adaptation and ambiguity, alternative market views, rolling horizons, and others.

Herein, we focus on the class of time-monotone forward performance criteria, studied in [58], where the performance criterion $U(x,t)$ is increasing and concave in x and decreasing in t. Such performance criteria have the following form:

$$U(x,t) = u(x, A_t)$$

where $u(x,t)$ is a deterministic function (cf. (2.5)) and A_t is the market input process (cf. (2.6)). Central to the construction of $u(x,t)$ is the choice of a finite positive Borel measure μ and its Laplace-type transform $h(x,t)$ given in (2.9). In [58], the authors also obtain closed form representations for the optimal wealth and the optimal portfolio processes given in (2.10) and (2.11).

We contribute to this line of research by establishing the long term behavior of the optimal wealth and optimal portfolio weights processes of such an investor. Our findings are twofold.

First, for the investor to avoid going bankruptcy eventually, the underlying measure μ of her forward performance must have a support with left-end less than 2. This is consistent with the long term behavior of the wealth of an investor with constant relative risk aversion (CRRA) in a log-normal market. We emphasize that such a condition on the measure μ does not prevent the investor from taking excessive risk. Rather, it allows the investor to de-leverage, or take less risk, if the market condition deems necessary.

Second, we find that in the long run, the investor adopts the investment strategy "closest" to the Kelly strategy within the bounds of her risk appetite. For example, if the investor uses a measure μ that implies that her relative risk tolerance coefficient is between 0.5 and 0.8, then in the long run, the investor will adopt a strategy corresponding to the one associated with constant relative risk tolerance equaling 0.8. If we further impose the condition that the relative risk tolerance coefficient is always less than 1, as evidenced by multiple theoretical and empirical studies in economics, then the investor adopts the riskiest strategy within the bounds of her risk tolerance in the long run.

The paper is structured as follows. In section 2, we present the market model, the investment performance criterion, and the closed-form representation of the optimal wealth and optimal portfolio weights processes. In section 3, we analyze the long term behavior of the optimal wealth process and its dependence on the underlying measure μ. We also compare our findings to that of a classical CRRA utility maximizer in a log-normal market setting. In section 4, we analyze the long term behavior of the optimal portfolio weights process and its relationship with the underlying measure μ.

2.2 The model and the investment performance criterion

The market environment consists of one riskless and k risky securities. The prices of the risky securities are modelled as Itô-diffusion processes, namely, the price S^i of the i^{th} risky asset follows

$$dS_t^i = S_t^i \left(\mu_t^i dt + \Sigma_{j=1}^d \sigma_t^{ji} dW_t^j \right),$$

with $S_0^i > 0$, for $i = 1, ..., k$. The process $W_t = \left(W_t^1, ..., W_t^d \right)$, $t \geq 0$, is a standard Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The coefficients μ_t^i and $\sigma_t^i = \left(\sigma^{1i}, ..., \sigma_t^{di} \right)$, $i = 1, ..., k$, $t \geq 0$, are $\mathcal{F}_t$-adapted processes and values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. We denote by σ_t the volatility matrix, i.e. the $d \times k$ random matrix $\left(\sigma_t^{ji} \right)$, whose i^{th} column represents the volatility σ_t^i of the i^{th} asset. We may, then, alternatively, write the above equation as

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i \cdot dW_t\right).$$

The riskless asset, the savings account, has price process B satisfying $dB_t = r_t B_t dt$ with $B_0 = 1$, and for a nonnegative $\mathcal{F}_t-$adapted interest rate process r_t. Also, we denote by μ_t the k-dimensional vector with coordinates μ_t^i and by $\mathbf{1}$ the k-dim vector with every component equal to one. The processes μ_t, σ_t and r_t satisfy the appropriate integrability conditions.

We assume that $\mu_t - r_t\mathbf{1} \in Lin\left(\sigma_t^T\right)$, where $Lin\left(\sigma_t^T\right)$ denotes the linear space generated by the columns of σ_t^T. Therefore, the equation $\sigma_t^T z = \mu_t - r_t\mathbf{1}$ has a solution, known as the market price of risk,

$$\lambda_t = \left(\sigma_t^T\right)^+ \left(\mu_t - r_t\mathbf{1}\right). \tag{2.1}$$

It is assumed that there exists a deterministic constant $c > 0$, such that $|\lambda_t| \leq c$ and that

$$\lim_{t\uparrow\infty} \int_0^t |\lambda_s|^2\, ds = \infty. \tag{2.2}$$

Starting at $t = 0$ with an initial endowment $x \geq 0$, the investor invests at any time $t > 0$ in the risky and riskless assets. The present value of the amounts invested are denoted by the processes π_t^0 and π_t^i, $i = 1, ..., k$, respectively, which are taken to be self-financing. The present value of her investment is then given by the discounted wealth process $X_t^\pi = \sum \pi_t^i$, $t > 0$, which solves

$$dX_t^\pi = \sigma_t \pi_t \cdot (\lambda_t dt + dW_t) \tag{2.3}$$

with the (column) vector $\pi_t = (\pi_t^i; i = 1, ..., k)$. It is taken to satisfy the non-negativity constraint $X_t^\pi \geq 0$, $t > 0$.

The set of admissible policies is given by

$$\mathcal{A} = \left\{ \pi : \text{self-financing},\ \ \pi_t \in \mathcal{F}_t,\ \ E_{\mathbb{P}} \int_0^t |\sigma_s \pi_s|^2 \, ds < \infty,\ \ X_t^\pi \geq 0,\ \ t > 0 \right\}.$$

The performance of admissible investment strategies is evaluated via the so-called forward investment performance criteria, introduced in [55] (see, also [56, 57, 59]). We review their definition next.

We introduce the domain notation $\mathbb{D}_+ = \mathbb{R}_+ \times [0, \infty)$ and $\mathbb{D} = \mathbb{R} \times [0, \infty)$.

Definition 2.2.1. *An $\mathcal{F}_t$-adapted process $U(x,t)$ is a forward investment performance if for $(x,t) \in \mathbb{D}$,*

i) the mapping $x \to U(x,t)$ is strictly increasing and strictly concave;

ii) for each $\pi \in \mathcal{A}$, $E_{\mathbb{P}}(U(X_t^\pi, t))^+ < \infty$, and for $s \geq t$,

$$U\left(X_t^\pi, t\right) \geq E_{\mathbb{P}}\left(U(X_s^\pi, s) \middle| \mathcal{F}_t \right),$$

iii) there exists $\pi^* \in \mathcal{A}$ such that for $s \geq t$,

$$U\left(X_t^{\pi^*}, t\right) = E_{\mathbb{P}}\left(U(X_s^{\pi^*}, s) \middle| \mathcal{F}_t \right).$$

Herein we focus on the class of *time-monotone* forward performance processes. For the reader's convenience, we rewrite some of the results we stated in the introduction. Time-monotone forward processes were extensively studied in [58], and are given by

$$U(x,t) = u(x, A_t), \tag{2.4}$$

where $u : \mathbb{D}_+ \to \mathbb{R}_+$ is strictly increasing and strictly concave in x, satisfying

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}}. \tag{2.5}$$

The market input processes A_t and M_t, $t \geq 0$, are defined as

$$M_t = \int_0^t \lambda_s \cdot dW_s \quad \text{and} \quad A_t = \int_0^t |\lambda_s|^2 \, ds = \langle M \rangle_t \, . \tag{2.6}$$

Central role in the construction of the performance criterion, the optimal policies and their wealth plays a harmonic function $h : \mathbb{D} \to \mathbb{R}_+$, defined via the transformation

$$u_x(h(z,t),t) = e^{-z+\frac{t}{2}}. \tag{2.7}$$

It solves, as it follows from (2.5) and (2.7), the ill-posed heat equation

$$h_t + \frac{1}{2}h_{zz} = 0. \tag{2.8}$$

Moreover, it is positive and strictly increasing in z. It was shown in [58], that such solutions are *uniquely* represented by

$$h(z,t) = \int_a^{+\infty} \frac{e^{yz-\frac{1}{2}y^2 t} - 1}{y} \nu(dy) + C,$$

where $a > 0$ and C a generic constant. The measure ν is defined on $\mathcal{B}^+(\mathbb{R})$, the set of positive Borel measures, with the additional properties that, for $z \in \mathbb{R}$,

$\int_a^{+\infty} e^{yz}\nu(dy) < \infty$ and $\int_a^{+\infty} \frac{1}{y}\nu(dy) < \infty$. To simplify the presentation and without loss of generality, we choose $C := \int_a^{+\infty} \frac{1}{y}\nu(dy)$ and, also, introduce the normalized measure $\mu\,(dy) = \frac{1}{y}\nu(dy)$.

Then, the function h has, for $(z,t) \in \mathbb{D}$, the representation

$$h(z,t) = \int_a^{+\infty} e^{yz-\frac{1}{2}y^2 t}\mu(dy) \quad \text{ with } \quad \int_a^{+\infty} \mu(dy) < \infty, \quad a > 0. \tag{2.9}$$

Moreover, it was shown in [58] that the optimal wealth process $X_t^{*,x}$ of an investor with initial wealth $x > 0$ under the forward performance criteria is given by

$$X_t^{*,x} = h\left(h^{(-1)}(x,0) + M_t + A_t, A_t\right). \tag{2.10}$$

The corresponding optimal portfolio process $\pi_t^{*,x}$ is given by

$$\pi_t^{*,x} = h_x\left(h^{(-1)}(x,0) + M_t + A_t, A_t\right)\sigma_t^{+}\lambda_t = r(X_t^{*,x}, A_t)\sigma_t^{+}\lambda_t, \tag{2.11}$$

where the (local) risk tolerance function $r\,(x,t) : \mathbb{D}_+ \to \mathbb{R}_+$ is defined as

$$r\,(x,t) := -\frac{u_x\,(x,t)}{u_{xx}\,(x,t)}. \tag{2.12}$$

From (2.7) and (2.12), we obtain that the risk tolerance function is represented as

$$r(x,t) = h_x\left(h^{(-1)}(x,t), t\right) = \int_a^{+\infty} y e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2 t}\mu(dy). \tag{2.13}$$

and the relative risk tolerance function

$$\hat{r}(x,t) = \frac{r(x,t)}{x} = \frac{\displaystyle\int_a^{+\infty} y e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2 t}\mu(dy)}{\displaystyle\int_a^{+\infty} e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2 t}\mu(dy)}. \tag{2.14}$$

Therefore, the *optimal portfolio weights* process $\hat{\pi}_t^{*,x}$ is given by

$$\hat{\pi}_t^{*,x} := \frac{\pi_t^{*,x}}{X_t^{*,x}} = \hat{r}(X_t^{*,x}, A_t)\sigma_t^{+}\lambda_t \tag{2.15}$$

where $\hat{r}$ is given in (2.14).

2.3 Long term behavior of optimal wealth process

Following (2.10), we can represent the optimal wealth process $X_t^{*,x}$ as follows.

$$\begin{aligned} X_t^{*,x} &= \int_a^{+\infty} e^{y(h^{(-1)}(x,0)+M_t+A_t)-\frac{1}{2}y^2A_t}\mu(dy) \\ &= \int_a^{+\infty} e^{y(M_t+A_t)-\frac{1}{2}y^2A_t}\mu_x(dy) \end{aligned} \tag{2.16}$$

where the measure μ_x is a finite positive Borel measure such that for any $A \in [a, +\infty)$,

$$\mu_x(A) = \int_A e^{yh^{(-1)}(x,0)}\mu(dy). \tag{2.17}$$

In particular, $\mu_x([a, +\infty)) = x$. Notice the integrand in (2.16) is the optimal wealth process of an investor with initial wealth \$1 and constant relative risk tolerance of y, and thus we denote it

$$X_t(\delta_y) := e^{y(M_t+A_t)-\frac{1}{2}y^2A_t} \tag{2.18}$$

Before we state the main theorem, we remind the reader of the definition of lower and upper class functions introduced by P. Leśy. Recall Blumenthal's 0-1 law implies that for any positive increasing continuous function $\phi(t)$, the probability

$$p := \mathbb{P}(W_t \leq \phi(t), t \uparrow \infty) \tag{2.19}$$

is either 0 or 1. Following P. Lévy [48], we say that

Definition 2.3.1. *An increasing function $\phi(t)$ belongs to the upper class (denoted by $\mathcal{U}$) if $p = 1$ and the lower class (denoted by $\mathcal{L}$) if $p = 0$, where p is given in* (2.19).

One can determine whether a function $\phi(t)$ belongs to the upper or lower class by resorting to the Kolmogorov's test (see Page 36 of [37]). For more details on the upper and lower classes and their connection to the law of iterated logarithms, we refer the readers to [24, 26, 37]. The following theorem connects the long term behavior of the optimal wealth process $X_t^{*,x}$ with the left-end of the support of the measure μ. In the case where the left-end equals 2, we show that the limiting behavior of $X_t^{*,x}$ hinges on whether the inverse function of a certain transform of the measure μ belongs to the lower or upper class.

Theorem 2.3.1. *Let a be the left-end of the support of the measure μ, and X_t^* be the optimal wealth process. Then, the following assertions hold.*

1. *If $a > 2$, then*
$$\lim_{t\uparrow\infty} X_t^* = 0, \ \ a.s. \tag{2.20}$$
2. *If $a < 2$, then*
$$\lim_{t\uparrow\infty} X_t^* = +\infty, \ \ a.s. \tag{2.21}$$
3. *In the case where $a = 2$, define the function $G : \mathbb{D} \to \mathbb{R}^+$ by*
$$G(z,t) := \int_2^{+\infty} e^{yz+y(1-\frac{y}{2})t} \mu_x(dy) \tag{2.22}$$

and $G^{(-1)}(z,t)$ its spatial inverse. The following dichotomy holds.

(a) If $G^{(-1)}(z,t) \in \mathcal{L}$ for all $z > 0$, then

$$\liminf_{t\uparrow\infty} X_t^* = 0 \quad \textit{and} \quad \limsup_{t\uparrow\infty} X_t^* = +\infty, \ \textit{a.s.} \tag{2.23}$$

(b) Otherwise, $G^{(-1)}(z,t) \in \mathfrak{U}$ for all $z > 0$, and (2.20) holds.

Proof. From (2.2), we have $\lim_{t\uparrow\infty} \frac{M_t}{A_t} = 0$, a.s. by the law of large numbers for local martingales. Therefore, we take $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for $\omega \in \Omega'$,

$$\lim_{t\uparrow\infty} A_t(\omega) = +\infty, \liminf_{t\uparrow\infty} M_t(\omega) = -\infty, \limsup_{t\uparrow\infty} M_t(\omega) = +\infty, \lim_{t\uparrow\infty} \frac{M_t}{A_t}(\omega) = 0. \tag{2.24}$$

In what follows, we let $\omega \in \Omega'$ and omit the ω-dependence in all processes to ease the notation. We have

$$\lim_{t\uparrow\infty} X_t(\delta_y) = \lim_{t\uparrow\infty} e^{yA_t(\frac{M_t}{A_t}+1-\frac{y}{2})} = \begin{cases} +\infty & \text{if } y < 2 \\ 0 & \text{if } y > 2. \end{cases} \tag{2.25}$$

For $y = 2$, $X_t(\delta_2) = e^{2M_t}$ and thus, from (2.24),

$$\limsup_{t\uparrow\infty} X_t(\delta_2) = +\infty \quad \text{and} \quad \liminf_{t\uparrow\infty} X_t(\delta_2) = 0 \tag{2.26}$$

We now present the long term behavior of X_t^* in three different cases.

(1) If $a > 2$, then for $y \in [a, +\infty)$ and $t > 0$, it is easy to check that $X_t(\delta_y) \leq X_t(\delta_a)$. Then, (2.20) follows from (2.25) and dominated convergence theorem.

(2) If $a < 2$, then it follows from (2.25) that, as $t \uparrow \infty$,

$$X_t^* \geq \int_a^{2-} X_t(\delta_y)\mu_x(dy) \to +\infty,$$

and we conclude with (2.21).

(3) In the case where $a = 2$, we first show that

$$\liminf_{t\uparrow\infty} X_t^* = 0. \tag{2.27}$$

To see this, notice that for $t > 0$,

$$0 \leq X_t^* \leq e^{2M_t}\mu_x([2, +\infty)) = xe^{2M_t}$$

and (2.27) follows from (2.24).

Next, we show that

$$\limsup_{t\uparrow\infty} X_t^* = \limsup_{t\uparrow\infty} G(W_t, t) \tag{2.28}$$

where G is defined in (2.22). To see this, notice that from (2.24), we have

$$\limsup_{t\uparrow\infty} X_t^* = \limsup_{t\uparrow\infty} G(M_t, A_t) = \limsup_{A_t\uparrow\infty} G(W_{A_t}, A_t) = \limsup_{t\uparrow\infty} G(W_t, t)$$

where we use the DDS theorem for $M_t = W_{A_t}$ in the second equality.

To show the dichotomy, suppose $G^{(-1)}(z_0, t) \in \mathcal{L}$ for some $z_0 > 0$. It follows from (2.19), $G(\cdot, t)$ being a strictly increasing function, and Definition 2.3.1 that

$$\mathbb{P}\left(\limsup_{t\uparrow\infty} G(W_t, t) > z_0\right) = \mathbb{P}\left(W_t > G^{(-1)}(z_0, t) \text{ infinitely often}\right) = 1. \tag{2.29}$$

Now we proceed to show that (2.29) is true for all $z > 0$ and hence, whether $G^{(-1)}(z,t)$ belongs to the lower or upper class is independent of the value of z. To see that, note for $z > 0$,

$$\frac{\partial}{\partial z} G^{(-1)}(z,t) = \frac{1}{G_z(G^{(-1)}(z,t),t)} = \frac{1}{\displaystyle\int_2^{+\infty} y e^{yG^{(-1)}(z,t)+y(1-\frac{y}{2})t}\mu_x(dy)} \leq \frac{1}{2z}.$$

Thus, for fixed $t > 0$ and $z_1 > 0$, we have

$$G^{(-1)}(z_1,t) - G^{(-1)}(z_0,t) = \int_{z_0}^{z_1} \frac{\partial}{\partial z} G^{(-1)}(z,t) dz \leq \int_{z_0}^{z_1} \frac{1}{2z} dz = \frac{1}{2} \ln\left(\frac{z_1}{z_0}\right).$$

It follows from (2.29) that for any $z_1 > 0$,

$$\limsup_{t\uparrow\infty} \frac{W_t}{G^{(-1)}(z_1,t)} \geq \limsup_{t\uparrow\infty} \frac{W_t}{G^{(-1)}(z_0,t) + \frac{1}{2}\ln(z_1/z_0)} = \limsup_{t\uparrow\infty} \frac{W_t}{G^{(-1)}(z_0,t)} > 1$$

Therefore,

$$\mathbb{P}\left(\limsup_{t\uparrow\infty} G(W_t,t) > z_1\right) = 1, \ \forall z_1 > 0 \tag{2.30}$$

and (2.23) follows from (2.27), (2.28), and (2.30).

Otherwise, $G^{(-1)}(z,t) \in \mathfrak{U}$ for all $z > 0$. In a similar fashion, we have

$$\mathbb{P}\left(\limsup_{t\uparrow\infty} G(W_t,t) \leq z\right) = \mathbb{P}\left(W_t \leq G^{(-1)}(z,t), t \uparrow \infty\right) = 1, \tag{2.31}$$

for all $z > 0$, and (2.20) follows from (2.27), (2.28), and (2.31). $\square$

The crux of the preceding theorem is a, the left-end of support of measure μ. It is clear from (2.14) that the relative risk tolerance function $\hat{r}(x,t) \geq a$ for any $x > 0, t \geq 0$. Therefore, a direct implication of Theorem 2.3.1 is that

If the investor's relative risk tolerance coefficient is *always* greater than 2, then her wealth will go down to 0 eventually.

This is not surprising considering the following analogous result in a log-normal market under the classical CRRA criteria.

In a market with one riskless asset of constant interest rate and one risky asset with constant volatility σ and Sharpe ratio λ. An investor with initial wealth $x > 0$ has a CRRA utility function at terminal time $T > 0$ given by

$$U(x) = \begin{cases} \dfrac{x^{1-\gamma} - 1}{1-\gamma}, & \gamma > 0 \text{ and } \gamma \neq 1 \\ \log x, & \gamma = 1 \end{cases}$$

Notice that the relative risk tolerance coefficient for such an investor is given by

$$-\frac{U'(x)}{xU''(x)} = \frac{1}{\gamma}.$$

It is well known the discounted optimal wealth at time $t > 0$ is given by

$$X_t^{*,x} = x \exp\left(\frac{1}{\gamma}(1 - \frac{1}{2\gamma})\lambda^2 t + \frac{1}{\gamma}\lambda W_t\right)$$

Therefore, we have the analagous long-term behavior of $X_t^{*,x}$

1. If $\frac{1}{\gamma} > 2$, then $\lim_{t\uparrow\infty} X_t^{*,x} = 0$, a.s.

2. If $\frac{1}{\gamma} < 2$, then $\lim_{t\uparrow\infty} X_t^{*,x} = +\infty$, a.s.

3. If the relative risk tolerance $\frac{1}{\gamma} = 2$, then $X_t^{*,x} = x\exp(2\lambda W_t)$, which satisfies

$$\limsup_{t\uparrow\infty} X_t^{*,x} = +\infty \text{ and } \liminf_{t\uparrow\infty} X_t^{*,x} = 0$$

The authors show in [30] that under the time-monotone forward performance criteria, the temporal limit of the relative risk tolerance function $\hat{r}(x,t)$ is exactly a, the left-end of the measure's support μ, i.e.

$$\lim_{t\uparrow\infty} \hat{r}(x,t) = a.$$

for any $x > 0$. This implies in the long run, a forward investor's risk preference is the same as that of an investor with constant relative risk tolerance a. Therefore, it is no surprise that the optimal wealth of these two investors share the same long term behavior.

2.4 Long term behavior of optimal portfolio weights process

Recall that the *optimal portfolio weights* process of an investor with initial wealth $x > 0$ is given in (2.15) as

$$\hat{\pi}_t^{*,x} := \frac{\pi_t^{*,x}}{X_t^{*,x}} = \hat{r}(X_t^{*,x}, A_t)\sigma_t^{+}\lambda_t,$$

and following (2.15) and (2.18), the *relative risk tolerance process* $\hat{r}(X_t^{*,x}, A_t)$ is given by

$$\hat{r}(X_t^{*,x}, A_t) = \frac{r(X_t^{*,x}, A_t)}{X_t^{*,x}} = \frac{\displaystyle\int_a^{+\infty} y X_t(\delta_y)\mu_x(dy)}{\displaystyle\int_a^{+\infty} X_t(\delta_y)\mu_x(dy)}. \tag{2.32}$$

where μ_x is the same as in (2.17). From (2.32), the relative risk tolerance process can be interpreted as the expected value of a continuum of relative risk tolerance coefficients in $[a, +\infty)$ with respect to a time-changing random probability measure.

The following theorem shows that in the long run, the investor places a fraction of her wealth in the tangency stock portfolio $\sigma_t^+ \lambda_t$ where the fraction is equal to the point in the support of measure μ closest to 1. In other words, the investor adopts the portfolio strategy "closest" to the renowned growth optimal, or Kelly, strategy [45] within the bounds of her risk appetite.

Theorem 2.4.1. *Suppose there is a unique c in the support of measure μ that is closest to 1, i.e.*

$$c := \arg\min_{y \in supp(\mu)} |y - 1|.$$

Then,

$$\lim_{t \uparrow \infty} \hat{r}(X_t^{*,x}, A_t) = c, \quad a.s. \tag{2.33}$$

and

$$\lim_{t \uparrow \infty} \frac{\hat{\pi}_t^{*,x}}{c \cdot \sigma_t^+ \lambda_t} = 1, \quad a.s. \tag{2.34}$$

Proof. As in the previous theorem, we take $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for $\omega \in \Omega'$,

$$\lim_{t \uparrow \infty} A_t(\omega) = +\infty, \liminf_{t \uparrow \infty} M_t(\omega) = -\infty, \limsup_{t \uparrow \infty} M_t(\omega) = +\infty, \lim_{t \uparrow \infty} \frac{M_t(\omega)}{A_t(\omega)} = 0. \tag{2.35}$$

First, note that (2.34) follows directly from (2.33) and (2.15). We now prove (2.33) in two different cases where $c \neq 1$ and $c = 1$, respectively. In what follows, we let $\omega \in \Omega'$ and omit the ω-dependence in all processes to ease the notation.

Case 1. Suppose $c \neq 1$. Without loss of generality, we suppose $c < 1$ and denote d the point in the support of μ to the right of 1 that is closet to 1, i.e.

$$d := \operatorname*{arg\,min}_{y \in \mathrm{supp}(\mu) \cap [1,+\infty)} |y-1|$$

and $0 < 1-c < d-1$. Then support of μ is a subset of $[a,c] \cup [d,+\infty)$. Let $\epsilon \in (0, \min\{1-c, \frac{d-1-(1-c)}{4}\})$. For t large enough, we have $|M_t/A_t| < \epsilon$, and observe that the fraction

$$\frac{X_t(\delta_y)}{X_t(\delta_{c-\epsilon})} = \exp\left((y-(c-\epsilon))A_t\left(\frac{M_t}{A_t} + 1 - \frac{y+c-\epsilon}{2}\right)\right)$$

has the following limits:

(1) For $y \in [a, c-\epsilon)$, $\frac{M_t}{A_t} + 1 - \frac{y+c-\epsilon}{2} > -\epsilon + 1 - (c-\epsilon) = 1-c$, and,

$$0 \leq \lim_{t\uparrow\infty} \frac{X_t(\delta_y)}{X_t(\delta_{c-\epsilon})} \leq \lim_{t\uparrow\infty} \exp\left((y-(c-\epsilon))A_t(1-c)\right) = 0;$$

(2) For $y \in (c-\epsilon, c]$, $\frac{M_t}{A_t} + 1 - \frac{y+c-\epsilon}{2} > -\epsilon + 1 - \frac{c+c-\epsilon}{2} = 1 - c - \frac{\epsilon}{2} > \frac{1-c}{2}$, and,

$$\lim_{t\uparrow\infty} \frac{X_t(\delta_y)}{X_t(\delta_{c-\epsilon})} \geq \lim_{t\uparrow\infty} \exp\left((y-(c-\epsilon))A_t \frac{1-c}{2}\right) = +\infty;$$

(3) For $y \in [d, +\infty)$, $\frac{M_t}{A_t} + 1 - \frac{y+c-\epsilon}{2} < \epsilon + 1 - \frac{d+c-\epsilon}{2} = \frac{3}{2}\epsilon - \frac{d-1-(1-c)}{2} < \frac{3}{2}\epsilon - 2\epsilon = -\frac{1}{2}\epsilon$, and,

$$0 \leq \lim_{t\uparrow\infty} \frac{X_t(\delta_y)}{X_t(\delta_{c-\epsilon})} \leq \lim_{t\uparrow\infty} \exp\left(-\frac{1}{2}\epsilon(y-(c-\epsilon))A_t\right) = 0.$$

It follows from dominated convergence theorem that

$$\lim_{t\uparrow\infty}\int_L \frac{X_t(\delta_y)}{X_t(\delta_{c-\epsilon})}\mu_x(dy) = \lim_{t\uparrow\infty}\int_L y\frac{X_t(\delta_y)}{X_t(\delta_{c-\epsilon})}\mu_x(dy)$$

$$= \begin{cases} 0 & \text{if } L=[a,c-\epsilon)\cup[d,+\infty) \\ \\ +\infty & \text{if } L=(c-\epsilon,c] \end{cases} \tag{2.36}$$

Thus, in the following expression,

$$\hat{r}(X_t^*,A_t) = \frac{\displaystyle\int_a^{+\infty} yX_t(\delta_y)\mu_x(dy)}{\displaystyle\int_a^{+\infty} X_t(\delta_y)\mu_x(dy)} = \frac{\displaystyle\int_a^{+\infty} y\frac{X_t(\delta_y)}{X_t(\delta_{c-\epsilon})}\mu_x(dy)}{\displaystyle\int_a^{+\infty} \frac{X_t(\delta_y)}{X_t(\delta_{c-\epsilon})}\mu_x(dy)}$$

$$= \frac{\left(\displaystyle\int_a^{(c-\epsilon)^+} + \int_{(c-\epsilon)^-}^{c} + \int_d^{+\infty}\right) y\frac{X_t(\delta_y)}{X_t(\delta_{c-\epsilon})}\mu_x(dy) + (c-\epsilon)\mu_x(\{c-\epsilon\})}{\left(\displaystyle\int_a^{(c-\epsilon)^+} + \int_{(c-\epsilon)^-}^{c} + \int_d^{+\infty}\right) \frac{X_t(\delta_y)}{X_t(\delta_{c-\epsilon})}\mu_x(dy) + \mu_x(\{c-\epsilon\})}$$

the integrals $\int_{(c-\epsilon)^-}^{c}$ dominate other terms in both the numerator and denominator. In addition,

$$c-\epsilon < \frac{\displaystyle\int_{(c-\epsilon)^-}^{c} y\frac{X_t(\delta_y)}{X_t(\delta_{c-\epsilon})}\mu_x(dy)}{\displaystyle\int_{(c-\epsilon)^-}^{c} \frac{X_t(\delta_y)}{X_t(\delta_{c-\epsilon})}\mu_x(dy)} \leq c.$$

Therefore, it follows that

$$c-\epsilon \leq \lim_{t\uparrow\infty}\hat{r}(X_t^*,A_t) \leq c,$$

and (2.33) follows as ϵ is arbitrary and $\mathbb{P}(\Omega')=1$.

Case 2. Suppose $c = 1$. Then $1 \in \text{supp}(\mu)$. Let $\epsilon > 0$, we must have $\mu((1-\epsilon, 1+\epsilon)) > 0$. For t large enough, we have $|M_t/A_t| < \frac{\epsilon}{2}$. For simplicity, denote the process

$$l_t = \frac{M_t}{A_t} + 1.$$

Observe that the fraction

$$\begin{aligned}\frac{X_t(\delta_y)}{X_t(\delta_{l_t-\epsilon})} &= \exp\left\{(y-(l_t-\epsilon))A_t\left(l_t - \frac{y+l_t-\epsilon}{2}\right)\right\}\\ &= \exp\left\{-\frac{1}{2}A_t(y-(l_t-\epsilon))(y-(l_t+\epsilon))\right\}\end{aligned}$$

has the following limits:

(1) For $y \in [a, l_t - \epsilon)$,

$$0 \leq \lim_{t\uparrow\infty} \frac{X_t(\delta_y)}{X_t(\delta_{l_t-\epsilon})} \leq \lim_{t\uparrow\infty} \exp\left\{\epsilon A_t(y-(l_t-\epsilon))\right\} = 0;$$

(2) For $y \in (l_t + \epsilon, +\infty)$,

$$0 \leq \lim_{t\uparrow\infty} \frac{X_t(\delta_y)}{X_t(\delta_{l_t-\epsilon})} \leq \lim_{t\uparrow\infty} \exp\left\{-\epsilon A_t(\omega)(y-(l_t(\omega)+\epsilon))\right\} = 0;$$

(3) For $y \in (l_t - \epsilon, l_t + \epsilon)$,

$$\lim_{t\uparrow\infty} \frac{X_t(\delta_y)}{X_t(\delta_{l_t-\epsilon})} = +\infty.$$

It follows from dominated convergence theorem that

$$\lim_{t\uparrow\infty} \int_L \frac{X_t(\delta_y)}{X_t(\delta_{l_t-\epsilon})}\mu_x(dy) = \lim_{t\uparrow\infty} \int_L y\frac{X_t(\delta_y)}{X_t(\delta_{l_t-\epsilon})}\mu_x(dy)$$

$$= \begin{cases} 0 & \text{if } L = [a, l_t - \epsilon) \cup (l_t + \epsilon, +\infty) \\ \\ +\infty & \text{if } L = (l_t - \epsilon, l_t + \epsilon) \end{cases} \tag{2.37}$$

Thus, in the following expression,

$$\hat{r}(X_t^*, A_t) = \frac{\displaystyle\int_a^{+\infty} y X_t(\delta_y)\mu_x(dy)}{\displaystyle\int_a^{+\infty} X_t(\delta_y)\mu_x(dy)} = \frac{\displaystyle\int_a^{+\infty} y \frac{X_t(\delta_y)}{X_t(\delta_{l_t-\epsilon})}\mu_x(dy)}{\displaystyle\int_a^{+\infty} \frac{X_t(\delta_y)}{X_t(\delta_{l_t-\epsilon})}\mu_x(dy)}$$

$$= \frac{\left(\displaystyle\int_a^{(l_t-\epsilon)^+} + \int_{(l_t-\epsilon)^-}^{(l_t+\epsilon)^+} + \int_{(l_t+\epsilon)^-}^{+\infty}\right) y \frac{X_t(\delta_y)}{X_t(\delta_{l_t-\epsilon})}\mu_x(dy) + (l_t \pm \epsilon)\mu_x(\{l_t \pm \epsilon\})}{\left(\displaystyle\int_a^{(l_t-\epsilon)^+} + \int_{(l_t-\epsilon)^-}^{(l_t+\epsilon)^+} + \int_{(l_t+\epsilon)^-}^{+\infty}\right) \frac{X_t(\delta_y)}{X_t(\delta_{l_t-\epsilon})}\mu_x(dy) + \mu_x(\{l_t \pm \epsilon\})}$$

the integrals $\int_{(l_t-\epsilon)^-}^{(l_t+\epsilon)^+}$ dominate other terms in both the numerator and denominator. In addition,

$$l_t - \epsilon < \frac{\displaystyle\int_{(l_t-\epsilon)^-}^{(l_t+\epsilon)^+} y \frac{X_t(\delta_y)}{X_t(\delta_{l_t-\epsilon})}\mu_x(dy)}{\displaystyle\int_{(l_t-\epsilon)^-}^{(l_t+\epsilon)^+} \frac{X_t(\delta_y)}{X_t(\delta_{l_t-\epsilon})}\mu_x(dy)} < l_t + \epsilon.$$

Therefore, it follow that

$$1 - \epsilon \leq \lim_{t\uparrow\infty} \hat{r}(X_t^*, A_t) \leq 1 + \epsilon$$

and (2.33) follows as $c = 1$, ϵ is arbitrary, and $\mathbb{P}(\Omega') = 1$.

$\square$

Theoretical and empirical studies in economics have shown that the coefficient of relative risk aversion, inverse of the relative risk aversion coefficient, is greater than 1 (see, among others, [5, 15, 17, 38, 62]). Therefore, it is safe to assume that the support of measure μ in (2.14) should be finite and the right-end b of the support of μ is no greater than 1 such that

$$\hat{r}(x,t) = \frac{\int_a^b y e^{yh^{(-1)}(x,t) - \frac{1}{2}y^2 t} \mu(dy)}{\int_a^b e^{yh^{(-1)}(x,t) - \frac{1}{2}y^2 t} \mu(dy)} \leq b \leq 1.$$

Under this assumption, we have the following corollary to Theorem 2.4.1.

Corollary 2.4.1. *Suppose the relative risk tolerance function $\hat{r}$ in (2.14) satisfies*

$$\hat{r}(x,t) \leq 1.$$

for all $x > 0, t \geq 0$. Then,

$$\lim_{t \uparrow \infty} \hat{r}(X_t^{*,x}, A_t) = b, \ \ a.s.$$

and

$$\lim_{t \uparrow \infty} \frac{\hat{\pi}_t^{*,x}}{b \cdot \sigma_t^+ \lambda_t} = 1, \ \ a.s.$$

where b is the right-end of the support of measure μ.

Chapter 3

Passive and competitive investment strategies under relative forward performance criteria

3.1 Introduction

This paper contributes to the study of the optimal portfolio decisions of two fund managers trading in an Itô-diffusion market when there are interactions between them. The type of interaction can be passive or competitive, and can be either with asset specialization or diversification.

In the passive case, one fund manager, say manager 1, takes the investment strategy of manager 2 as given, and is concerned with the relative performance of her portfolio. Thus, instead of her absolute wealth, X_t^1, she considers a relative performance metric, denoted by $\tilde{X}_t^1$, which weighs her wealth by a power of the wealth of manager 2, X_t^2. Specifically, the relative performance metric of manager 1 is defined as

$$\tilde{X}_t^1 := \frac{X_t^1}{(X_t^2)^{\theta_1}},$$

with $\theta_1 \in [0,1]$. The parameter θ_1 measures the intensity of manager $1'$s relative performance bias.

If $\theta_1 = 0$, she does not take into account the actions of manager 2 at all, and the investment problem reduces to the classical one. When $\theta_1 = 1$, then the process X_t^2 resembles a traditional benchmark. Besides taking into account this arbitrary, but fixed, strategy of fund manager 2, manager 1 does not interact with him at all. Rather, she solves in isolation an investment problem with a modified "wealth" process. We call this case *passive*.

There are two sub-cases of passive relative performance. The first is when each fund manager trades between a common riskless asset and an exclusive stock. This is the case of *asset specialization*. The two stocks are proxies for two distinct asset classes. Asset specialization occurs frequently and reflects a well-documented phenomenon that fund managers invest in assets with which they are familiar (see, among others, [6, 19, 40, 52, 53, 68, 69, 70]). These works attribute asset specialization to a variety of reasons, with familiarity with a certain sector, reduction of costs to enhance knowledge of new stocks, trading costs and constraints, liquidity costs and ambiguity aversion being some of them.

By far, the most important feature when combining relative performance and asset specialization is that because of imperfect correlation, each manager faces an incomplete market problem, parametrized by the exogenously chosen policy of the other manager.

The second sub-case happens when both fund managers trade in the same market environment, that is, among the riskless asset and the two representative stocks. This is the case of *asset diversification*. It occurs among peers trading in the same asset class. Frequently, the motivation is to increase money flows (see, among others, [9, 21]). In a different direction, asset diversification also occurs in the delegated portfolio management, where the role of one manager is replaced by the client (see, among others, [43]). In a related family of models, they also appear in the so-called "Catching up with the Joneses" literature (see [1, 32]).

An alternative direction is when the two fund managers compete with each other, in that the investment choices of one affect the other. Then, both investors solve, under relative performance criteria, interactive optimal portfolio problems by simultaneously adjusting their policies in response to the actions of the other manager. We call this case of interlinked asset management *competitive*. A natural concept of optimality is the one of a Nash equilibrium.

Competition is well documented in investment practice (see, for example, [2, 16, 28, 31, 67]). As in the passive investment direction, there are two sub-cases, the asset specialization and the diversification ones.

The above cases were recently concisely studied in [8] under the assumptions that both fund managers have power utilities, but with different risk aversions, and trade in log-normal markets for the same terminal horizon. The main objective therein is to investigate the link between competition and asset specialization.

Due to the homogeneity of the risk preferences and the log-normality assumptions, explicit solutions are obtained for all cases. A detailed qualitative analysis is, in turn, provided using the ranges of the fund managers' risk tolerance coefficients and their bias parameters, the stocks' correlation and their Sharpe ratios.

Herein, we use the work of [8] as a starting point to study more extended settings by considering Itô-diffusion markets and flexible investment horizons. The motivation for the latter comes from the fact that fund managers do not act in a single trading horizon but in rolling ones, from one evaluation period to the next. While one may argue that once an evaluation period finishes, an entirely new investment problem can be defined, this is not what happens in practice, for past performance is taken into account and is not discarded.

We cast the above four problems, namely, with relative passive and competitive performance concerns, and with and without asset specialization, in a forward performance setting. Such criteria, were introduced in [55, 56] and subsequently generalized in [57, 58, 59].

Forward criteria allow for flexible model revision and investment horizons, which are ubiquitous elements in dynamic asset allocation. They are built on the classical Dynamic Programming Principle (DPP) and define a process, the forward performance one, with the following properties: compiled with the wealth process generated by any admissible strategy, the forward process is always a (local) supermartingale; there exists an optimal policy that generates a wealth process when compiled with the forward process yields a (local) martingale.

We generalize the original definition of forward performance to accommodate both cases, the one with passive relative performance and the one under competition. For the latter case, we define the concept of a forward Nash equilibrium.

For the passive cases, with and without asset specialization, we focus on forward criteria that are locally riskless processes, in that their volatility coefficient is taken to be zero for all times. This case has been examined in detail in [58] and provides the simplest case of a forward criterion in general Itô-diffusion markets. Such a criterion has the interesting property of remaining time-decreasing for all investment horizons.

When relative performance criteria are involved, however, time monotonicity is not always valid in the asset specialization case. It holds, however, in the diversification case. For both cases, we derive stochastic PDEs that the forward criteria are expected to solve and show how the (local) supermartingality and martingality properties are related to the solutions of these SPDE. Naturally, the SPDE for the two cases are distinct, reflecting mainly the market incompleteness of the former market setting. For the latter case, we argue that, as the relative performance bias parameter θ_1 converges to 1, the forward SPDE converges to the one solved in the benchmark case in [57]. In this case, the forward criterion can be explicitly represented in terms of a deterministic function compiled with a stochastic market input. When $\theta_1 \neq 1$, however, such functional representation is not valid in general and path dependence emerges. This is demonstrated in the CRRA examples we provide, which are solved explicitly.

For the forward Nash equilibrium, the analogous SPDE is supposed to solved a system of ill-posed HJB equations. The derivation and study of this system is left for future research. For both competitive cases, we provide explicit examples for CRRA criteria.

The paper is structured as follows. In section 2, we introduce the model and examine the asset specialization setting. We consider the passive and the competitive cases, and introduce, respectively, definitions of the relative forward criterion and the forward Nash equilibrium. We present some general results and a concrete example for the CRRA criteria. In section 3, we examine the diversification (no asset specialization) setting, also for the passive and competitive cases. In analogy to the results of section 2, we introduce the relative forward criterion and the forward Nash equilibrium, provide some general results and present the CRRA case. We conclude in section 4 with a discussion of future research directions.

3.2 Asset specialization and forward relative performance criteria

We consider a market environment with one riskless asset, and two risky securities, which serve as proxies of two distinct asset classes. We assume their prices, S_t^1 and S_t^2, are Itô-diffusions solving

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i dW_t^i \right), \tag{3.1}$$

with $S_0^i > 0$, $i = 1, 2$.

The processes W_t^1 and W_t^2 are standard Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by (W^1, W^2). Their correlation is $\rho \in (-1, 1)$.

The market coefficients μ_t^i and σ_t^i, $i = 1, 2$, $t > 0$ are $\mathcal{F}_t$-adapted processes with values in $\mathbb{R}^+$ and $\mathbb{R}$, respectively. The riskless asset, a discount bond, has an $\mathcal{F}_t$-adapted interest rate process $r_t > 0$, and serves as the numeraire.

The related market price of risk processes are given by

$$\lambda_t^i = \frac{\mu_t^i - r_t}{\sigma_t^i}, \tag{3.2}$$

$i = 1, 2$, and it is assumed that for all $t > 0$, $|\lambda_t^i| \leq C$, with C a positive constant.

In this market, two portfolio managers, indexed by $i = 1, 2$, trade between the riskless and the risky securities, *specializing* in assets 1 and 2, respectively. Without loss of generality, both managers' initial endowments are taken to be the same, denoted by $x > 0$.

Their portfolio strategies, π_t^1 and π_t^2, $t \geq 0$, denote the fractions of fund invested in the risky asset by managers 1 and 2, respectively. They are taken to be self-financing which together with (3.1) yield that the (discounted) wealths of manager 1 and 2 solve

$$\frac{dX_t^i}{X_t^i} = \sigma_t^i \pi_t^i \left(\lambda_t^i dt + dW_t^i\right) \tag{3.3}$$

for $i = 1, 2$. The set of admissible policies $\mathcal{A}^i$ for manager i is defined as

$$\mathcal{A}^i = \left\{ \pi^i : \text{self-financing and } \pi^i_t \in \mathcal{F}_t, \mathbb{P}(\int_0^t (\sigma^i_s \pi^i_s X^i_s)^2 ds < \infty) = 1, X^i_t > 0, \forall t > 0 \right\}. \tag{3.4}$$

Each manager measures the performance of her strategy taking into account the policy of the other. Specifically, let us assume that manager 2 follows an admissible strategy $\pi^2 \in \mathcal{A}^2$, which generates a wealth X^2_t, $t > 0$, as in (3.3). Then, the *relative performance metric* of manager 1 is defined to be

$$\tilde{X}^1_t := \frac{X^1_t}{(X^2_t)^{\theta_1}}, \tag{3.5}$$

with X^1_t solving (3.3). The parameter $\theta_1 \in [0, 1]$ reflects her *relative performance bias*, that is the extent to which she biases her objectives towards relative performance concerns.

When $\theta_1 = 0$, the fund manager is not concerned about relative performance at all. Then, the portfolio choice problem reduces to the classical one, in that manager 1 trades on her own without taking into consideration any investment strategy that manager 2 might follow.

Respectively, if manager 1 follows a strategy $\pi^1 \in \mathcal{A}^1$, the performance metric of manager 2 is defined as

$$\tilde{X}^2_t := \frac{X^2_t}{(X^1_t)^{\theta_2}}, \tag{3.6}$$

with the analogous relative performance bias parameter $\theta_2 \in [0, 1]$.

We then see that the processes $(X^2_t)^{\theta_1}$ and $(X^1_t)^{\theta_2}$ can be viewed as *benchmarks* for manager 1 and 2, respectively. We stress however that, contrary to the classical settings in which the benchmarks are traded processes, this is not the case herein due to the asset specialization and imperfect correlation assumptions. Therefore, each manager faces an *incomplete* market investment problem.

The above criteria are in accordance with the ones proposed in [8] in a log-normal market. Namely, if we define the relative return processes $\tilde{R}_t^i, i = 1, 2$, as

$$\tilde{R}_t^1 := \frac{X_t^1}{X_t^2} \quad \text{and} \quad \tilde{R}_t^2 := \frac{X_t^2}{X_t^1},$$

with X^1, X^2 in (3.3), then the relative performance metrics defined in (3.5) and (3.6) can be written as

$$\tilde{X}_t^1 = (X_t^1)^{1-\theta_1}(\tilde{R}_t^1)^{\theta_1} \quad \text{and} \quad \tilde{X}_t^2 = (X_t^2)^{1-\theta_2}(\tilde{R}_t^2)^{\theta_2}.$$

These are the metrics proposed on page 8 of [8].

Herein we consider a general Itô-diffusion market model, introduce the concept of relative performance criteria for each fund manager, and find the associated optimal policies and their wealth processes. As we discussed in the introduction, we consider two separate cases.

In the first case, the relative performance criterion is *passive* in the sense that the policy, with regards to which performance is measured, is taken to be arbitrary but fixed. There is no other interaction between the two managers and, thus, each manager solves an individual investment problem "in isolation".

In the second case, there is a dynamic, *competitive* interaction between the two managers. The action of each manager is not taken as fixed anymore.

Instead, they continuously adjust their investment policies to the policy of each other.

As in [8], we assume that both managers have full information of all fundamentals of the investment environment, even though they do not invest in both assets. This is an idealized case, partially supported by the results in [46], where the author argues that each manager can infer these fundamentals from the other manager's return, which is observable in practice. We comment on this limitation in section 4.

Given the distinct nature of the above cases –passive and competitive– we introduce two new concepts, the *forward passive* and the *forward competitive performance criteria*. We will also frequently refer to the latter as the *forward Nash criterion*.

3.2.1 Forward relative performance criteria

Each manager takes an investment strategy of the other as given, and optimizes, for all times $t > 0$, the output of her relative performance metric. To measure this output, we introduce a performance criterion, the forward relative one, modeled as an $\mathcal{F}_t$-adapted process $U^i(x,t)$ for $i = 1, 2$.

This new criterion is a variation of the original one, proposed by Musiela and Zariphopoulou (see [55, 56]) and further developed by them and others (see, [44, 57, 59, 58, 60, 61, 73]).

The main idea for the forward relative performance criterion is as follows: if manager 2 follows an investment policy, say π_t^2, $t > 0$, then, for any strategy $\pi^1 \in \mathcal{A}^1$, the process $U^1(\tilde{X}^1_s, s)$ is a (local) supermartingale and, furthermore, there exists $\pi^{1,*} \in \mathcal{A}^1$ such that $U^1(\tilde{X}^{1,*}_s, s)$ is a (local) martingale. The processes $\tilde{X}^1_s$ and $\tilde{X}^{1,*}_s$ solve (3.5), with π^1_t and $\pi^{1,*}_t$ used, respectively. The analogous conditions hold for manager 2.

As expected, this relative criterion is implicitly and exogenously parametrized by the generic policy π^2 of the manager 2. We stress, however, that the policy of manager 2 is taken to be arbitrary but fixed, without assuming any optimality properties of the policy for manager 2.

We introduce this criterion next. With a slight abuse of notation, we use x to denote the state process $\tilde{X}^i$, $i = 1, 2$, given by (3.5) and (3.6) (and not X^i as in (3.3)).

Definition 3.2.1. *Let $\pi^2 \in \mathcal{A}^2$ (resp. $\pi^1 \in \mathcal{A}^1$) be an arbitrary but fixed admissible policy of manager 2 (resp. manager 1). An $\mathcal{F}_t$-adapted process $U^1(x,t)$ (resp. $U^2(x,t)$) is a forward relative performance for manager 1 (resp. manager 2) if, for $t \geq 0$ and $x > 0$, the following conditions hold:*

i) The mappings $x \to U^i(x,t)$, $i = 1, 2$, are strictly increasing and strictly concave.

ii) For each $\pi^1 \in \mathcal{A}^1$, $U^1(\tilde{X}^1_t, t)$ is a local supermartingale, where $\tilde{X}^1$ is the relative performance metric given in (3.5). Respectively, for each $\pi^2 \in \mathcal{A}^2$, $U^2\left(\tilde{X}^2_t, t\right)$ is a local supermartingale, where $\tilde{X}^2$ is the relative performance metric given in (3.6).

iii) There exists $\pi^{1,*} \in \mathcal{A}^1$ such that $U^1\left(\tilde{X}^{1,*}_t, t\right)$ is a local martingale, *where* $\tilde{X}^{1,*}$ *as in (3.5) with* $\pi^{1,*}$ *being used. Respectively, there exists* $\pi^{2,*} \in \mathcal{A}^2$ *such that* $U^2\left(\tilde{X}_t^{2,*}, t\right)$ *is a local martingale, where* $\tilde{X}^{2,*}$ *as in (3.6) with* $\pi^{2,*}$ *being used.*

In the above definition, we do not make explicit references to the initial conditions $U^1(x, 0)$ and $U^2(x, 0)$, but we assume that admissible initial data exist such that the above definition is viable. The specification of the class of admissible initial conditions is a challenging task (see, for example, [10, 58]) - even in the absence of relative performance concerns - and is currently being investigated by the authors.

In general, forward performance criteria (relative or not) are Itô-diffusion processes with non-zero volatility coefficient. Contrary to the classical expected utility case, the forward volatility process is an investor-specific input. Once it is chosen, the supermartingality and martingality properties impose conditions on the drift of the process. Under enough regularity, these conditions lead to the forward performance SPDE (see [59]). Furthermore, depending on whether the forward process is path-dependent or a deterministic functional of stochastic factors, the forward volatility can be path- or state-dependent (see, for example, [41, 44, 49, 60, 61, 66]).

Herein, we focus on the simplest possible case, the one of zero volatility. Such processes are extensively analyzed in [58] in the absence of relative performance concerns. Therein, a concise characterization of the forward criteria is given together with necessary and sufficient conditions for their existence and uniqueness. In that setting, the zero-volatility forward processes are always time-decreasing processes. However, this is *not* the case when *relative* performance is considered (see (3.13) and (3.21)).

Next we derive a stochastic PDE for smooth relative performance criteria of zero volatility. Namely, we assume that their Itô decomposition is of the form $dU^i(x,t) = U_t^i(x,t)\,dt,\ i = 1,2$, and that the derivatives $U_t^i(x,t), U_x^i(x,t)$ and $U_{xx}^i(x,t)$ exist a.s. for $t \geq 0$.

To ease the presentation, we only state the results for fund manager 1, since the symmetric ones hold for manager 2.

Proposition 3.2.1. *Assume that manager* 2 *follows policy* $\pi_t^2 \in \mathcal{A}^2$. *Consider the stochastic PDE*

$$U_t^1 = \frac{1}{2}\left(\lambda_t^1 - \rho\theta_1\sigma_t^2\pi_t^2\right)^2 \frac{(U_x^1)^2}{U_{xx}^1}$$
$$+\,\theta_1\sigma_t^2\pi_t^2\left(\lambda_t^2 - \frac{1+\theta_1}{2}\sigma_t^2\pi_t^2 - \rho(\lambda_t^1 - \rho\theta_1\sigma_t^2\pi_t^2)\right)xU_x^1 - \frac{1-\rho^2}{2}(\theta_1\sigma_t^2\pi_t^2)^2x^2U_{xx}^1, \tag{3.7}$$

and assume that, for an admissible initial condition $U^1(x,0)$, *it has a smooth solution* $U^1(x,t)$, $t > 0$, *such that the mapping* $x \to U^1(x,t)$ *is strictly increasing and strictly concave in* x, *for each* $t > 0$.

Let the portfolio strategy $\pi_t^{1,*}, t > 0$, *defined by*

$$\pi_t^{1,*} = R_t^{1,*}\frac{\lambda_t^1}{\sigma_t^1} + \rho\theta_1\left(1 - R_t^{1,*}\right)\frac{\sigma_t^2}{\sigma_t^1}\pi_t^2, \tag{3.8}$$

with

$$R_t^{1,*} := -\frac{U_x^1(\tilde{X}_t^{1,*},t)}{\tilde{X}_t^{1,*}U_{xx}^1(\tilde{X}_t^{1,*},t)}, \tag{3.9}$$

where $\tilde{X}_t^{1,*}$ *solves* (3.10) *with* $\pi_t^{1,*}$ *being used. If, for* $t > 0$, $\pi_t^{1,*} \in \mathcal{A}^1$ *and* $\tilde{X}_t^{1,*}$ *is well defined, then* $U^1(x,t)$ *is a forward relative performance process. Furthermore, the above policy* $\pi_t^{1,*}$, $t > 0$ *is optimal.*

Proof. Applying Itô's rule to (3.5) yields

$$\frac{d\tilde{X}_t^1}{\tilde{X}_t^1} = \left(\lambda_t^1 \sigma_t^1 \pi_t^1 + \theta_1 \sigma_t^2 \pi_t^2 (\frac{1+\theta_1}{2} \sigma_t^2 \pi_t^2 - \lambda_t^2) - \rho \theta_1 \sigma_t^1 \sigma_t^2 \pi_t^1 \pi_t^2 \right) dt$$
$$+ \sigma_t^1 \pi_t^1 dW_t^1 - \theta_1 \sigma_t^2 \pi_t^2 dW_t^2. \tag{3.10}$$

Let $U^1(x,t)$ be a solution to (3.7) with the aforementioned properties and $\tilde{X}_t^1$ as in (3.5). Then, applying the Itô-Wentzell formula (see [47]) to $U^1(\tilde{X}_t^1, t)$ gives

$$dU^1(\tilde{X}_t^1, t) = U_t^1(\tilde{X}_t^1, t)dt + U_x^1(\tilde{X}_t^1, t)d\tilde{X}_t^1 + \frac{1}{2} U_{xx}^1(\tilde{X}_t^1, t) d\langle \tilde{X}^1 \rangle_t$$
$$= U_x^1(\tilde{X}_t^1, t)\tilde{X}_t^1 \sigma_t^1 \pi_t^1 dW_t^1 - U_x^1(\tilde{X}_t^1, t)\tilde{X}_t^1 \theta_1 \sigma_t^2 \pi_t^2 dW_t^2$$
$$+ \left(U_t^1(\tilde{X}_t^1, t) + U_x^1(\tilde{X}_t^1, t)\tilde{X}_t^1 \left(\lambda_t^1 \sigma_t^1 \pi_t^1 + \theta_1 \sigma_t^2 \pi_t^2 (\frac{1+\theta_1}{2} \sigma_t^2 \pi_2^2 - \lambda_t^2) - \rho \theta_1 \sigma_t^1 \sigma_t^2 \pi_t^1 \pi_t^2 \right) \right.$$
$$\left. + \frac{1}{2} U_{xx}^1(\tilde{X}_t^1, t)(\tilde{X}_t^1)^2 \left((\sigma_t^1 \pi_t^1)^2 + (\theta_1 \sigma_t^2 \pi_t^2)^2 - 2\rho \theta_1 \sigma_t^1 \sigma_t^2 \pi_t^1 \pi_t^2 \right) \right) dt.$$

Using that $U^1(x,t)$ solves (3.7), the above becomes

$$dU^1(\tilde{X}_t^1, t) = U_x^1(\tilde{X}_t^1, t)\tilde{X}_t^1 \sigma_t^1 \pi_t^1 dW_t^1 - U_x^1(\tilde{X}_t^1, t)\tilde{X}_t^1 \theta_1 \sigma_t^2 \pi_t^2 dW_t^2$$
$$+ \frac{1}{2} (\tilde{X}_t^1)^2 U_{xx}^1(\tilde{X}_t^1, t) \left| \sigma_t^1 \pi_t^1 - \rho \theta_1 \sigma_t^2 \pi_t^2 + \frac{(\lambda_t^1 - \rho \theta_1 \sigma_t^2 \pi_t^2) U_x^1(\tilde{X}_t^1, t)}{\tilde{X}_t^1 U_{xx}^1(\tilde{X}_t^1, t)} \right|^2 dt. \tag{3.11}$$

The concavity assumption of $U^1(x,t)$ implies that the drift term above is non-positive, and vanishes when

$$\sigma_t^1 \pi_t^{1,*} = (\lambda_t^1 - \rho \theta_1 \sigma_t^2 \pi_t^2) \left(- \frac{U_x^1(\tilde{X}_t^{1,*}, t)}{\tilde{X}_t^{1,*} U_{xx}^1(\tilde{X}_t^{1,*}, t)} \right) + \rho \theta_1 \sigma_t^2 \pi_t^2$$

$$= \lambda_t^1 R_t^{1,*} + \rho\theta_1 \left(1 - R_t^{1,*}\right) \sigma_t^2 \pi_t^2,$$

with $R_t^{1,*}$ as in (3.9), and (3.8) follows.

Thus, if $\pi_t^{1,*} \in \mathcal{A}^1$ and the associated wealth process $X_t^{1,*}$ is well-defined, the process $U^1(\tilde{X}^{1,*}, t)$ is a local martingale, otherwise it is a local supermartingale. We easily conclude. $\square$

The optimal portfolio $\pi_t^{1,*}$ is a linear combination of a myopic-type one, $R_t^{1,*} \frac{\lambda_t^1}{\sigma_t^1}$, and a strategy that is proportional to the one followed by manager 2. The proportionality process depends linearly on the correlation ρ, the relative performance bias parameter θ_1, the ratio of the two volatilities, and the factor $\left(1 - R_t^{1,*}\right)$.

The process $R_t^{1,*}$ is the relative risk tolerance given in (3.9). As it is case in the classical portfolio setting, the cases $R_t^{1,*} \lesseqgtr 1$ render different behavior. Indeed, we can see, for example, that if both ρ and π^2 are positive, investing under relative performance criteria yields policies bigger (resp. smaller) than the myopic-type term if $R_t^{1,*} \leq 1$ (resp. $R_t^{1,*} \geq 1$).

Note also that if $\rho = 0$, then $\pi_t^{1,*} = R_t^{1,*} \frac{\lambda_t^1}{\sigma_t^1}$. However, while π^2 does not explicitly appear in the form of $\pi_t^{1,*}$, its effects are present in the metric $\tilde{X}_t^{1,*}$ which enters in the definition of $R_t^{1,*}$.

3.2.1.1 The CRRA case

We provide an explicitly solved example in which fund manager 1 starts with a power criterion. There is no assumption on the criterion that the manager 2 uses, as we only use an admissible policy π^2 chosen by her.

Proposition 3.2.2. *Suppose manager* 1 *starts at* $t = 0$ *with a power criterion of the form,*

$$U^1(x,0) = \frac{1}{1-\gamma_1} x^{1-\gamma_1}, \tag{3.12}$$

with $\gamma_1 > 0, \gamma_1 \neq 1$. *Also, assume that manager 2 follows an admissible portfolio strategy* π_t^2, $t > 0$. *Then, the following assertions hold:*

i) The relative performance criterion for manager 1 *is given by*

$$U^1(x,t) = \frac{1}{1-\gamma_1} x^{1-\gamma_1} \exp\left(- \int_0^t \frac{1}{2} \frac{1-\gamma_1}{\gamma_1} \eta_s^1 ds \right), \tag{3.13}$$

where

$$\eta_t^1 := \left(\lambda_t^1 - \delta_t^1 \theta_1 \sigma_t^2 \pi_t^2\right)^2 + \left(\rho^2 (1-\gamma_1)^2 + \gamma_1 (1 - \gamma_1 + \frac{1}{\theta_1}) - \left(\delta_t^1\right)^2 \right) (\theta_1 \sigma_t^2 \pi_t^2)^2, \tag{3.14}$$

with

$$\delta_t^1 := \gamma_1 \frac{\lambda_t^2}{\lambda_t^1} + \rho(1-\gamma_1). \tag{3.15}$$

ii) The associated optimal portfolio $\pi_t^{1,*}$ *is given by*

$$\pi_t^{1,*} = \frac{1}{\gamma_1} \frac{\lambda_t^1}{\sigma_t^1} + \rho \theta_1 \left(1 - \frac{1}{\gamma_1}\right) \frac{\sigma_t^2}{\sigma_t^1} \pi_t^2. \tag{3.16}$$

Proof. We look for a criterion in the separable form

$$U^1(x,t) = \frac{x^{1-\gamma_1}}{1-\gamma_1} K_t,$$

with $K_0 = 1$ and K_t being an $\mathcal{F}_t$-adapted process, differentiable in t. Using equation (3.7), we deduce that

$$\frac{1}{1-\gamma} \frac{dK_t}{dt} + q_t K_t = 0,$$

where

$$q_t := \frac{1}{2\gamma_1}(\lambda_t^1 - \rho\theta_1\sigma_t^2\pi_t^2)^2$$

$$-\theta_1\sigma_t^2\pi_t^2\left(\lambda_t^2 - \frac{\theta_1+1}{2}\sigma_t^2\pi_t^2 - \rho(\lambda_t^1 - \rho\theta_1\sigma_t^2\pi_t^2)\right) - \frac{1-\rho^2}{2}\gamma_1(\theta_1\sigma_t^2\pi_t^2)^2$$

$$= \frac{1}{2\gamma_1}\left(\left(\lambda_t^1 - \theta_1\delta_t^1\sigma_t^2\pi_t^2\right)^2 + \left(\rho^2(1-\gamma_1)^2 + \gamma_1(1-\gamma_1+\frac{1}{\theta_1}) - \left(\delta_t^1\right)^2\right)(\theta_1\sigma_t^2\pi_t^2)^2\right), \tag{3.17}$$

with δ_t^1 as in (3.15). Direct calculations then yield (3.13) and (3.14). In turn, using (3.8), we deduce (3.16). □

We easily recover the analogous quantities for fund manager 2, listed below for completeness and future reference. We have,

$$U^2(x,t) = \frac{1}{1-\gamma_2}x^{1-\gamma_2}\exp\left(-\int_0^t \frac{1}{2}\frac{1-\gamma_2}{\gamma_2}\eta_s^2 ds\right), \tag{3.18}$$

where

$$\eta_t^2 := \left(\lambda_t^2 - \delta_t^2\theta_2\sigma_t^1\pi_t^1\right)^2 + \left(\rho^2(1-\gamma_2)^2 + \gamma_2(1-\gamma_2+\frac{1}{\theta_2}) - \left(\delta_t^2\right)^2\right)(\theta_2\sigma_t^1\pi_t^1)^2, \tag{3.19}$$

with $\delta_t^2 := \gamma_2\frac{\lambda_t^1}{\lambda_t^2} + \rho(1-\gamma_2)$. The associated optimal portfolio $\pi_t^{2,*}$ is given by

$$\pi_t^{2,*} = \frac{1}{\gamma_2}\frac{\lambda_t^2}{\sigma_t^2} + \rho\theta_2\left(1-\frac{1}{\gamma_2}\right)\frac{\sigma_t^1}{\sigma_t^2}\pi_t^1. \tag{3.20}$$

When $\gamma_1 = 1$, the initial forward process is logarithmic, $U^1(x,0) = \log x$. Direct calculations then yield that a relative forward logarithmic criterion is given by the process

$$U^1(x,t) = \log x + K_t, \tag{3.21}$$

where

$$\frac{dK_t}{dt} = -\frac{1}{2}(\lambda_s^1 - \theta_1 \sigma_s^2 \pi_s^2)^2 - \theta_1 \sigma_s^2 \pi_s^2 (\lambda_s^1 - \lambda_s^2), \tag{3.22}$$

and $K_0 = 0$. Therefore,

$$U^1(x,t) = \log x - \int_0^t \left(\frac{1}{2}(\lambda_s^1 - \theta_1 \sigma_s^2 \pi_s^2)^2 + (\lambda_s^1 - \lambda_s^2)\theta_1 \sigma_s^2 \pi_s^2 \right) ds.$$

The optimal control $\pi_t^{1,*}$ is the myopic portfolio λ_t^1/σ_t^1.

As we mentioned earlier, it is not always the case that under passive relative performance concerns, the zero-volatility criteria are time-monotone. Indeed, this can be seen from (3.17) and (3.22) where the time derivatives of the process K_t do *not* have a fixed sign.

3.2.2 Forward Nash equilibrium

We now consider the case that the two fund managers compete with each other. Namely, if manager 2 uses policy π^2, then manager 1 optimizes his performance responding to this strategy, and manager 2 does, in turn, the same. To measure the performance of their interactive policies, we propose the concept of forward Nash equilibrium, which we introduce next.

For the reader's convenience, we will use the notation

$$\tilde{X}_t^1(\pi_2^*) = \frac{X_t^1}{(X_t^{2,*})^{\theta_1}} \quad \text{and} \quad \tilde{X}_t^2(\pi_1^*) = \frac{X_t^2}{(X_t^{1,*})^{\theta_2}} \tag{3.23}$$

where X_t^1, X_t^2 solve (3.3) for $\pi^1 \in \mathcal{A}^1$ and $\pi^2 \in \mathcal{A}^2$, and $X_t^{1,*}$, $X_t^{2,*}$ solve (3.3) with the candidate Nash policies $\pi_t^{1,*}, \pi_t^{2,*}$ used, respectively.

Definition 3.2.2. *A forward Nash equilibrium consists of two pairs of $\mathcal{F}_t$-adapted processes, $(U^1(x,t), \pi_t^{1,*})$ and $(U^2(x,t), \pi_t^{2,*})$, such that, for $t \geq 0$, the following assertions hold:*

i) $\pi^{1,*} \in \mathcal{A}^1$ and $\pi^{2,*} \in \mathcal{A}^2$.

ii) For any $\pi^1 \in \mathcal{A}^1$ and $\pi^2 \in \mathcal{A}^2$, the processes $U^1(\tilde{X}_t^1(\pi_2^*), t)$ and $U^2(\tilde{X}_t^2(\pi_1^*), t)$ are local supermartingales, with the processes $\tilde{X}_t^1(\pi_2^*)$ and $\tilde{X}_t^2(\pi_1^*)$ given by (3.23).

iii) The processes $U^1(\tilde{X}_t^{1,*}(\pi_2^*), t)$ and $U^2(\tilde{X}_t^{2,*}(\pi_1^*), t)$ are local martingales, where

$$\tilde{X}_t^{1,*}(\pi_2^*) = \frac{X_t^{1,*}}{(X_t^{2,*})^{\theta_1}} \qquad \text{and} \qquad \tilde{X}_t^{2,*}(\pi_1^*) = \frac{X_t^{2,*}}{(X_t^{1,*})^{\theta_2}}.$$

If, under appropriate integrability conditions, the processes $U^1(\tilde{X}_t^1(\pi_2^*), t)$ and $U^2(\tilde{X}_t^2(\pi_1^*), t)$, and $U^1(\tilde{X}_t^{1,*}(\pi_2^*), t)$ and $U^2(\tilde{X}_t^{2,*}(\pi_1^*), t)$ are, respectively, supermartingales and martingales, then, for $t \geq 0$, we have the following results. For all $\pi_1 \in \mathcal{A}_1$,

$$\mathbb{E}\left(U^1(\tilde{X}_t^{1,*}(\pi_2^*), t)\right) = \mathbb{E}\left(U^1(x^{1-\theta_1}, 0)\right) \geq \mathbb{E}\left(U^1(\tilde{X}_t^1(\pi_2^*), t)\right).$$

Similarly, for all $\pi_2 \in \mathcal{A}_2$,

$$\mathbb{E}\left(U^2(\tilde{X}_t^{2,*}(\pi_1^*), t)\right) = \mathbb{E}\left(U^2(x^{1-\theta_2}, 0)\right) \geq \mathbb{E}\left(U^2(\tilde{X}_t^2(\pi_1^*), t)\right).$$

In other words, no unilateral deviation in strategy by either manager will result in an increase in the expected utility of her relative performance metric.

The specification of the forward Nash equilibrium appears to be intractable for general forward criteria, for one needs to solve the system of equations

$$\begin{cases} \pi_t^{1,*} = R_t^{1,*}\dfrac{\lambda_t^1}{\sigma_t^1} + \rho\theta_1\left(1 - R_t^{1,*}\right)\dfrac{\sigma_t^2}{\sigma_t^1}\pi_t^{2,*} \\ \\ \pi_t^{2,*} = R_t^{2,*}\dfrac{\lambda_t^2}{\sigma_t^2} + \rho\theta_2\left(1 - R_t^{2,*}\right)\dfrac{\sigma_t^1}{\sigma_t^2}\pi_t^{1,*} \end{cases}$$

with

$$R_t^{1,*} := -\frac{U_x^1(\tilde{X}_t^{1,*},t)}{\tilde{X}_t^{1,*}U_{xx}^1(\tilde{X}_t^{1,*},t)} \quad \text{and} \quad R_t^{2,*} := -\frac{U_x^2(\tilde{X}_t^{2,*},t)}{\tilde{X}_t^{2,*}U_{xx}^2(\tilde{X}_t^{2,*},t)}.$$

However, this is in general not feasible since both controlled processes $\tilde{X}_t^{1,*}, \tilde{X}_t^{2,*}$, entering in $R_t^{1,*}, R_t^{2,*}$, depend on $\pi_t^{1,*}$ and $\pi_t^{2,*}$, respectively.

3.2.2.1 The CRRA case

We assume that managers 1 and 2 start at $t = 0$ with initial criteria of the form

$$U^i(x,0) = \frac{1}{1-\gamma_i}x^{1-\gamma_i}$$

for $x > 0$, $\gamma_i > 0$, and $\gamma_i \neq 1$, $i = 1, 2$.

The result below yields an explicit construction of a forward Nash equilibrium under asset specialization.

Proposition 3.2.3. *i) Let $\rho \neq 0$. Under asset specialization, a forward Nash equilibrium is given by the pairs $(U^1(x,t), \pi_t^{1,*})$ and $(U^1(x,t), \pi_t^{2,*})$, where*

$$U^1(x,t) = \frac{1}{1-\gamma_1}x^{1-\gamma_1}\exp\left(-\int_0^t \frac{1}{2}\frac{1-\gamma_1}{\gamma_1}\eta_s^{1,*}ds\right) \tag{3.24}$$

where $\eta_s^{1,*}$*is as in (3.14) with* $\pi_t^{2,*}$ *used instead of* π_t^2*. The optimal portfolio strategy of manager 1 is given by*

$$\pi_t^{1,*} = \frac{1}{\sigma_t^1}\left(\frac{\gamma_2\lambda_t^1}{\gamma_1\gamma_2 - \rho^2\theta_1\theta_2(\gamma_1-1)(\gamma_2-1)} + \rho\theta_1\frac{(\gamma_1-1)\lambda_t^2}{\gamma_1\gamma_2 - \rho^2\theta_1\theta_2(\gamma_1-1)(\gamma_2-1)}\right). \tag{3.25}$$

Respectively,

$$U^2(x,t) = \frac{1}{1-\gamma_2}x^{1-\gamma_2}\exp\left(-\int_0^t \frac{1}{2}\frac{1-\gamma_2}{\gamma_2}\eta_s^{2,*}ds\right) \tag{3.26}$$

with $\eta_s^{2,*}$ *is as in (3.19) with* $\pi_t^{1,*}$ *used instead of* π_t^1*. The optimal portfolio strategy of manager 2 is given by*

$$\pi_t^{2,*} = \frac{1}{\sigma_t^2}\left(\frac{\gamma_1\lambda_t^2}{\gamma_1\gamma_2 - \rho^2\theta_1\theta_2(\gamma_1-1)(\gamma_2-1)} + \rho\theta_2\frac{(\gamma_2-1)\lambda_t^1}{\gamma_1\gamma_2 - \rho^2\theta_1\theta_2(\gamma_1-1)(\gamma_2-1)}\right). \tag{3.27}$$

ii) If $\rho = 0$*, then*

$$U^1(x,t) = \frac{1}{1-\gamma_1}x^{1-\gamma_1}\exp\left(-\int_0^t \frac{1}{2}\frac{1-\gamma_1}{\gamma_1}\eta_s^1 ds\right)$$

with $\eta_1 = \left(\lambda_t^1 - \theta_1\rho(1-\gamma_1)\frac{1}{\gamma_2}\lambda_t^2\right)^2 + ((1-\gamma_1)\theta_1 - 1)\frac{\gamma_1}{\gamma_2}\theta_1(\lambda_t^2)^2$*, and*

$$\pi_t^{1,*} = \frac{1}{\gamma_1}\frac{\lambda_t^1}{\sigma_t^1}.$$

Furthermore,

$$U^2(x,t) = \frac{1}{1-\gamma_2}x^{1-\gamma_2}\exp\left(-\int_0^t \frac{1}{2}\frac{1-\gamma_2}{\gamma_2}\eta_s^2 ds\right)$$

with $\eta_2 = \left(\lambda_t^2 - \theta_2\rho(1-\gamma_2)\frac{1}{\gamma_1}\lambda_t^1\right)^2 + ((1-\gamma_2)\theta_2 - 1)\frac{\gamma_2}{\gamma_1}\theta_2(\lambda_t^1)^2$*, and*

$$\pi_t^{2,*} = \frac{1}{\gamma_2}\frac{\lambda_t^2}{\sigma_t^2}.$$

Proof. At the forward Nash equilibrium, we see from Proposition 3.2.2 that both optimal equilibrium strategies $\pi^{1,*}, \pi^{2,*}$ should satisfy the system of equations

$$\begin{cases} \pi_t^{1,*} - \left(1 - \dfrac{1}{\gamma_1}\right) \rho\theta_1 \dfrac{\sigma_t^2}{\sigma_t^1} \pi_t^{2,*} = \dfrac{1}{\gamma_1} \dfrac{\lambda_t^1}{\sigma_t^1} \\ \\ -\left(1 - \dfrac{1}{\gamma_2}\right) \rho\theta_2 \dfrac{\sigma_t^1}{\sigma_t^2} \pi_t^{1,*} + \pi_t^{2,*} = \dfrac{1}{\gamma_2} \dfrac{\lambda_t^2}{\sigma_t^2}. \end{cases}$$

If $\rho = 0$, the above equations simplify to $\pi_t^{1,*} = \frac{1}{\gamma_1} \frac{\lambda_t^1}{\sigma_t^1}$ and $\pi_t^{2,*} = \frac{1}{\gamma_2} \frac{\lambda_t^2}{\sigma_t^2}$. If $\rho \neq 0$, the solution $\left(\pi_t^{1,*}, \pi_t^{2,*}\right)$ is well defined and given by (3.25) and (3.27), since

$$\det \begin{vmatrix} 1 & -\left(1 - \frac{1}{\gamma_1}\right) \rho\theta_1 \frac{\sigma_t^2}{\sigma_t^1} \\ -\left(1 - \frac{1}{\gamma_2}\right) \rho\theta_2 \frac{\sigma_t^1}{\sigma_t^2} & 1 \end{vmatrix} \neq 0,$$

as, for $i = 1, 2$, $\theta_i \in [0, 1]$ and $1 - \frac{1}{\gamma_i} < 1$. In turn, using (3.13) and (3.18) for the appropriate auxiliary policies $\eta_t^{1,*}$ and $\eta_t^{2,*}$we conclude. □

3.3 Diversification (no asset specialization) and relative forward performance criteria

We now revert our attention to the case where the two managers invest in the same market, namely, they trade among the common riskless asset and both stocks. As in the asset specialization case, they are concerned with the their performance relative to the other manager. In analogy, we have the passive and the competitive cases, as they were defined in sections 3.3.1 and 3.3.2.

While there are various similarities between the asset specialization and the diversification settings under relative performance concerns, the fundamental difference is that in the former case each fund manager faces an incomplete market problem while in the latter each faces a complete one. Therefore, under diversification, the relative concerns can be directly related to optimizing in terms of a benchmark or a traded security. However, because the relative performance bias parameters θ_i, $i = 1, 2$ may be strictly less than 1, certain scaling properties fail, and it may not be feasible to reduce the problem to a traditional investment problem with benchmarking.

3.3.1 Passive forward performance criteria

In this section, both managers trade in both asset classes with representative prices S_t^1, S_t^2, $t > 0$, as in (3.1). To ease the presentation of the wealth and performance metric processes, we use a different parametrization of price dynamics, setting

$$W_t := \begin{bmatrix} W_t^1 \\ -\frac{\rho}{\sqrt{1-\rho^2}} W_t^1 + \frac{1}{\sqrt{1-\rho^2}} W_t^2 \end{bmatrix}$$

where W^1, W^2 are the correlated Brownian motions in the last section. Furthermore, we represent the volatility matrix σ_t and the market price of risk vector λ_t as

$$\sigma_t = \begin{bmatrix} \sigma_t^1 & \rho\sigma_t^2 \\ 0 & \sqrt{1-\rho^2}\sigma_t^2 \end{bmatrix} \qquad \text{and} \qquad \lambda_t = \begin{bmatrix} \lambda_t^1 \\ -\frac{\rho}{\sqrt{1-\rho^2}} \lambda_t^1 + \frac{1}{\sqrt{1-\rho^2}} \lambda_t^2 \end{bmatrix}$$

Then, the (discounted) wealth processes X_t^1, X_t^2 can be written

$$\frac{dX_t^1}{X_t^1} = \sigma_t \pi_t^1 \cdot (\lambda_t dt + dW_t) \quad \text{and} \quad \frac{dX_t^2}{X_t^2} = \sigma_t \pi_t^2 \cdot (\lambda_t dt + dW_t). \tag{3.28}$$

where the vector π_t^1 (resp. π_t^2) denotes the fraction of wealth manager 1 (resp. manager 2) invest in both asset classes S^1 and S^2.

The sets of admissible policies $\mathcal{A}^i$, $i = 1, 2$ are defined similarly to their counterpart in (3.4), using now as filtration $(\mathcal{F}_t^W)_{t \geq 0}$ the one generated by W_t. As in the asset specialization case, we define the performance metrics

$$\tilde{X}_t^1 := \frac{X_t^1}{(X_t^2)^{\theta_1}} \quad \text{ and } \quad \tilde{X}_t^2 := \frac{X_t^2}{(X_t^1)^{\theta_2}} \tag{3.29}$$

with X_t^1, X_t^2 solving (3.28) and the relative performance bias parameters $\theta_1, \theta_2 \in [0, 1]$.

We also analogously define the passive relative forward performance criteria $U^i(x, t)$, $i = 1, 2$, adjusting Definition 3.2.1 appropriately for the wealth processes in (3.28) (instead of (3.3)), and the relative performance metrics in (3.29) (instead of (3.5) and (3.6)). To ease the presentation, we do not provide the new definition, since the modification is straightforward. Furthermore, we also focus on forward criteria that have zero volatility, and assume that the derivatives $U_t^i(x, t), U_x^i(x, t)$ and $U_{xx}^i(x, t)$ exist a.s. for all $t \geq 0$.

Next, we provide a stochastic PDE that the (passive) relative performance process of fund manager 1 satisfies. The arguments are similar to the ones in Proposition 3.2.1, and are therefore omitted.

Contrary to the asset specialization case, the process $U^1(x, t)$ is, for each $x > 0$, *always* time-decreasing. This follows from the form of (3.30), $\theta_1 \in [0, 1]$, and the spatial strict concavity and strict monotonicity assumptions for $U^1(x, t)$.

Proposition 3.3.1. *Assume that manager 2 uses policy $\pi_t^2 \in \mathcal{A}^2$. Consider the stochastic PDE*

$$U_t^1 = \frac{1}{2}\left|\lambda_t - \theta_1 \sigma_t \pi_t^2\right|^2 \frac{\left(U_x^1\right)^2}{U_{xx}^1} - \frac{\theta_1(1-\theta_1)}{2}\left|\sigma_t \pi_t^2\right|^2 x U_x^1, \tag{3.30}$$

and assume that it has a smooth solution, with an admissible initial datum $U^1(x,0)$, and such that the mapping $x \to U^1(x,t)$ is strictly increasing and strictly concave in x, for each $t > 0$.

Let the portfolio strategy $\pi_t^{1,*}$ defined by

$$\pi_t^{1,*} = r_t^* \sigma_t^{-1} \lambda_t + (1 - r_t^*)\, \theta_1 \pi_t^2 \tag{3.31}$$

with r_t^*

$$r_t^* := -\frac{U_x^1(\tilde{X}_t^{1,*}, t)}{\tilde{X}_t^{1,*} U_{xx}^1(\tilde{X}_t^{1,*}, t)},$$

where $\tilde{X}_t^{1,*}$ solves (3.28) with $\pi_t^{1,*}$ used.

If $\pi_t^{1,*} \in \mathcal{A}^1$ and the solution $\tilde{X}_t^{1,*}, t > 0$, is well defined, then the process $U^1(x,t)$ is a time-decreasing relative forward performance for manager 1. Furthermore, the above policy $\pi_t^{1,*}$, $t > 0$, is optimal.

The optimal portfolio $\pi_t^{1,*}$consists of two components. The first is of myopic-type, equals $r_t^* \sigma_t^{-1} \lambda_t$. The second is proportional to the policy π_t^2. The proportionality process depends on the parameter θ_1 and the process $1 - r_t^*$, where r_t^* is the optimal relative risk tolerance. As in the asset specialization case, the cases $r_t^* > 1$ (resp. $r_t^* < 1$) are expected to render different signs if the sign of π^2 is kept constant.

A special case is when $\theta_1 = 1$. Then, the wealth process X_t^2, generated by the admissible policy π^2, can be identified with the benchmark process Y_t in page 164 of [57]. In this case, the SPDE (3.30) reduces to

$$U_t^1 = \frac{1}{2}\left|\lambda_t - \sigma_t \pi_t^2\right|^2 \frac{\left(U_x^1\right)^2}{U_{xx}^1}.$$

Following the results therein, we deduce that $U^1(x,t)$ is uniquely represented as

$$U^1(x,t) = u^1\left(x, \int_0^t |\lambda_2 - \sigma_s \pi_s^2|^2 ds\right),$$

where $u^1 : \mathbb{R}^+ \times [0,\infty) \longrightarrow \mathbb{R}^+$ satisfies the deterministic constraint $u_t^1 = \frac{1}{2}\frac{\left(u_x^1\right)^2}{u_{xx}^1}$. For the solutions of this pde, we refer the reader to [58]. If, however, $\theta_1 \neq 1$, the above reduction is not feasible, except for some special cases. Analogous results can be derived for the investment problem of manager 2.

3.3.1.1 The CRRA case

We provide an explicitly solved example in which fund manager 1 starts with a power criterion. As in the asset specialization case, there is no assumption on the criterion that the other manager uses, as we only use an admissible policy π^2 chosen by her.

Proposition 3.3.2. *Suppose that manager* 1 *starts at* $t = 0$ *with a power criterion of the form*

$$U^1(x,0) = \frac{1}{1-\gamma_1} x^{1-\gamma_1},$$

with $\gamma_1 > 0, \gamma_1 \neq 1$. *Also assume that manager 2 follows an admissible portfolio strategy* $\pi_t^2, t > 0$. *Then, the following assertions hold:*

i) A relative performance criterion for manager 1 *is given by*

$$U^1(x,t) = \frac{1}{1-\gamma_1} x^{1-\gamma_1} \exp\left(- \int_0^t \frac{1}{2} \frac{1-\gamma_1}{\gamma_1} \eta_s ds \right), \tag{3.32}$$

where

$$\eta_t^1 = \left| \lambda_t - \theta_1 \sigma_t \pi_t^2 \right|^2 + \gamma_1 \left(\frac{1}{\theta_1} - 1 \right) \left| \theta_1 \sigma_t \pi_t^2 \right|^2.$$

ii) The associated optimal portfolio $\pi_t^{1,*}$ *is given by*

$$\pi_t^{1,*} = \frac{1}{\gamma_1} \sigma_t^{-1} \lambda_t + \left(1 - \frac{1}{\gamma_1} \right) \theta_1 \pi_t^2. \tag{3.33}$$

When $\gamma_1 = 1$, the logarithmic relative forward criterion is given by the process

$$U^1(x,t) = \log x - \frac{1}{2} \int_0^t \left(|\lambda_s - \theta_1 \sigma_s \pi_s^2|^2 + \theta_1 (1-\theta_1) |\sigma_s \pi_s^2|^2 \right) ds.$$

The optimal control $\pi_t^{1,*}$ is then the myopic portfolio $\sigma_t^{-1} \lambda_t$.

Note that contrary to the asset specialization case (cf. (3.14)), the process $\eta_t^1 > 0$, $t > 0$. If $\gamma_1 < 1$, then $U^1(x,t) > 0$ and the time exponential is decreasing. The reverse properties hold when $\gamma_1 > 1$. In both cases, for each $x > 0$, $U^1(x,t)$ is time-decreasing as argued earlier. As in the power case, for each $x > 0$, the above logarithmic process is also time-decreasing.

3.3.2 Forward Nash equilibrium

The forward Nash equilibrium is defined as in Definition 3.2.2 with the only difference being that the wealth processes $X_t^1, X_t^{1,*}$ and X_t^2, $X_t^{2,*}$ now solve (3.28) with the the appropriate analogous investment strategies.

As in the asset specialization case, the general case is intractable, due to the interlinked dependencies among the individual forward performance processes, the wealth processes and the candidate forward Nash portfolios. An explicit solution is constructed for initial criteria of power type next.

3.3.2.1 The CRRA case

We assume that managers 1 and 2 start at $t = 0$ with initial criteria of the form

$$U^i(x, 0) = \frac{1}{1-\gamma_i} x^{1-\gamma_i}$$

for $x > 0$, $\gamma_i > 0$, and $\gamma_i \neq 1$, $i = 1, 2$.

Proposition 3.3.3. *Define the constants*

$$c_1 = \frac{\gamma_1 - \theta_2(1-\gamma_2)}{\gamma_1\gamma_2 - \theta_1\theta_2(1-\gamma_1)(1-\gamma_2)}, \quad c_2 = \frac{\gamma_2 - \theta_1(1-\gamma_1)}{\gamma_1\gamma_2 - \theta_1\theta_2(1-\gamma_1)(1-\gamma_2)},$$

and

$$C_1 = (1-\theta_1 c_1)^2 + \gamma_1\theta_1(1-\theta_1)c_1^2, \qquad C_2 = (1-\theta_2 c_2)^2 + \gamma_2\theta_2(1-\theta_2)c_2^2.$$

Under asset diversification, a forward Nash equilibrium is given by the pairs $(U^1(x,t), \pi_t^{1,*})$ *and* $(U^1(x,t), \pi_t^{2,*})$*, where*

$$U^i(x,t) = \frac{1}{1-\gamma_i} x^{1-\gamma_i} \exp\left(-\int_0^t \frac{1}{2}\frac{1-\gamma_i}{\gamma_i} C_i |\lambda_s|^2 ds\right)$$

and

$$\pi_t^{i,*} = c_i \sigma_t^{-1} \lambda_t.$$

for $i = 1, 2$.

Proof. At the Nash equilibrium, we see from the results in Proposition 3.3.1 that both optimal portfolio strategies $(\pi_t^{1,*}, \pi_t^{2,*})$ should satisfy the system of equations

$$\begin{cases} \pi_t^{1,*} - \left(1 - \dfrac{1}{\gamma_1}\right)\theta_1 \pi_t^{2,*} = \dfrac{1}{\gamma_1}\sigma_t^{-1}\lambda_t \\ \\ -\left(1 - \dfrac{1}{\gamma_2}\right)\theta_2 \pi_t^{1,*} + \pi_t^{2,*} = \dfrac{1}{\gamma_2}\sigma_t^{-1}\lambda_t. \end{cases}$$

After tedious but routine calculations, we conclude. □

The forward Nash equilibria strategies $\pi_t^{1,*}$ and $\pi_t^{2,*}$ are proportional to the myopic portfolio $\sigma_t^{-1}\lambda_t$. The proportionality constants c_1, c_2 depend only on $\gamma_1, \gamma_2, \theta_1$ and θ_2. Note that $\pi_t^{1,*}$ depends on c_2 while $\pi_t^{2,*}$ on c_1.

3.4 Extensions

In all cases herein, we have assumed that model selection, trading and relative performance valuation are all aligned and, furthermore, that they occur continuously in time. In reality, these three fundamental attributes are not synchronized. A more realistic scenario is when trading takes place more frequently than model selection, and relative performance evaluation takes place less frequently than trading. The extreme case is the classical expected utility problem in which the utility is specified only once, at initial time. With regards to the relative frequency of trading and model selection, it is more realistic to assume that the model is selected for some trading period ahead, say a week, and within this week, trading takes place in discrete or continuous time.

When relative performance is involved, the distinct scales of time evolution are more critical, for each fund manager announces her performance at discrete times and never continuously.

To incorporate such phenomena in the forward setting, one needs to work with more general classes of forward performance processes. A recent effort has appeared in [4] where the concept of predictable forward performance processes is proposed and an explicit example is provided for a binomial model with adaptive market parameter selection. One may extend these results incorporating relative performance, which however will lead to an incomplete case that may pose several difficulties.

Another issue is the type of information, under asset specialization, each fund manager may obtain about each other, namely, her risk aversion, relative bias, and the stock dynamics. The results in [46] give some partial answers to how one manager may infer this information from the realized, and publicly available, returns of the other manager. However, several "under-specification" issues remain open. It might be more realistic to assume that at the end of each relative evaluation period, each fund manager receives information about the performance of the other and, right after, formulates a view about the possible upcoming performance till the end of the next evaluation period. This will partially address the absence of complete information under asset specialization. In this case, injecting personal views could lead to a forward Black-Litterman type criterion under competition.

Chapter 4

Modeling realized beta time series using high-frequency intra-day asset prices

4.1 Introduction

In the single-factor capital asset pricing model (CAPM), the systematic risk of an asset return, beta, is determined by its covariance with the market return normalized by the market variance (see, among others, [50, 51, 64]). In this paper, we investigate the question whether a stock's beta is constant over time. If not, an important question is how the time series characteristics of the moving beta influence our modeling decisions. Specifically, we are interested in examining the dynamics in beta and modeling beta utilizing high-frequency intraday asset price data.

An implicit consensus among economists is that betas are indeed time-varying (see, for example, [34]). Even in the absence of explicit allowance for non-constant betas, the CAPM is typically implemented using estimation windows, usually five to ten years (see, among others, [13, 25]). The same is true for online financial data services. For example, Yahoo finance utilizes the monthly stock returns in the past three years to calculate the beta value of a stock, while Google Finance and Bloomberg rely on a five-year estimation window. Furthermore, theoretical and empirical studies in asset pricing are often carried out in conditional frameworks to allow for time-varying parameters.

We proceed as follows. In section 2 we establish the theoretical framework on which our subsequent statistical analysis is based. In section 3 we conduct some preliminary analysis on the time series plots of beta and some related variables. Specifically, we investigate the persistent nature by computing two related statistics and observing the correlograms. In section 4, drawing insights from [7], we move beyond the point estimate of the true beta and establish the confidence intervals dependent on the sampling frequency of our model. In section 5, given our results from the previous sections, we recast beta in a simple dynamic linear model.

4.2 Theoretical framework

We use the notations in [7]. Suppose the logarithmic price process, $y(t)$, is a N-dimensional continuous Itô process with the dynamics

$$dy(t) = \mu(t)dt + \Omega(t)dW(t)$$

where $W(t)$ is a standard N-dimensional Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The coefficients μ and Ω are $\mathcal{F}(t)$-adapted processes satisfying the appropriate integrability conditions. We treat the Nth element of $y(t)$ as the log price of the market and the nth element the log price of the ith individual stock included in the analysis. Similarly, the covariance matrix Ω contains both the market variance, $\Omega_{N,N}$, and the individual covariance with the market, $\Omega_{n,N}$.

Next, consider a fixed interval of time of length $\hbar > 0$. For concreteness we typically refer to $\hbar$ as representing a day. Traditional daily returns are computed as

$$r_i = y(i\hbar) - y((i-1)\hbar), \quad i = 1, 2, \ldots$$

where i indexes the day. However, our focus will be on the case where we additionally have M equally spaced intra-$\hbar$ high frequency observations during each $\hbar$ time period. The jth intra-$\hbar$ return for the ith period (e.g., if $\hbar$ is a day and $M = 1440$, then this is the return for the jth minute on the ith day) will be calculated as

$$r_{j,i} := y\left((i-1)\hbar + \frac{\hbar j}{M}\right) - y\left((i-1)\hbar + \frac{\hbar(j-1)}{M}\right), \quad j = 1 \ldots M$$

High frequency returns allow us to compute the *realized covariance matrix* for the ith day

$$[y_M]_i = \sum_{j=1}^{M} r_{j,i} r'_{j,i}$$

The notation $[y_M]_i$ is designed to reflect that this matrix is based on the y process using M intra-$\hbar$ observations and computed on the ith day. The reason for the use of the square brackets will become clearer in a moment when we recall the idea of quadratic variation. The realized covariation matrix is clearly different from the empirical covariance matrix of high frequency returns

$$\frac{1}{M}\sum_{j=1}^{M} r_{j,i} r'_{j,i} - \frac{1}{M^2}\left(\sum_{j=1}^{M} r_{j,i}\right)\left(\sum_{j=1}^{M} r_{j,i}\right)' = \frac{1}{M}\sum_{j=1}^{M} r_{j,i} r'_{j,i} - \frac{1}{M^2} r_i r'_i,$$

which converges to a matrix of zeros in probability as $M \to \infty$. Recall the definition of the quadratic variation (QV) of $y(t)$ is defined as

$$\langle y \rangle_t = p \lim_{M\to\infty} \sum_{j=0}^{M-1} \{y(t_{j+1}) - y(t_j)\}\{y(t_{j+1}) - y(t_j)\}'$$

for any sequence $0 = t_0 < \cdots < t_M = t$ with $\sup_j\{t_{j+1} - t_j\} \to 0$. This implies that the realized covariance matrix

$$[y_M]_i = \sum_{j=1}^{M} r_{j,i} r'_{j,i} \xrightarrow{\mathbb{P}} \langle y \rangle_i = \langle y \rangle_{\hbar i} - \langle y \rangle_{\hbar(i-1)} = \int_{\hbar(i-1)}^{\hbar i} \Omega'(t)\Omega(t)dt$$

as $M \to \infty$, meaning realized covariation, $[y_M]_i$, consistently estimates increments of QV, $\langle y \rangle_i$.

In other words, by summing up sufficiently finely-sampled high-frequency returns, it is possible to construct ex-post *realized volatility measures* for the integrated latent volatilities that are asymptotically free of measurement errors. This contrasts sharply with the common use of the cross-product of the $\hbar$-period returns, $r_i \cdot r'_i$ as a simple ex-post variability measure. Although the squared return over the forecast horizon provides an unbiased estimate for the integrated volatility, it is an extremely noisy estimator, and predictable variation in the true latent volatility process is typically dwarfed by measurement error. Moreover, for longer horizons any conditional mean dependence will tend to contaminate this variance measure.

We now denote the realized market volatility by

$$\hat{v}^2_{m,i} = \sum_{j=1}^{M} r^2_{(N),j,i},$$

and we denote the realized covariance between the market and the ith individual stock return by

$$\hat{v}_{nm,i} = \sum_{j=1}^{M} r_{(n),j,i} \cdot r_{(N),j,i}.$$

We then define the associated realized beta as

$$\hat{\beta}_{n,i} = \frac{\hat{v}_{nm,i}}{\hat{v}_{m,i}^2}.$$

It is straightforward to check that the realized beta consistently estimates the underlying integrated beta in the following sense:

$$\hat{\beta}_{n,i} \xrightarrow{\mathbb{P}} \beta_{n,i} = \frac{\displaystyle\int_{\hbar(i-1)}^{\hbar i} \Sigma_{(nN)}(u)du}{\displaystyle\int_{\hbar(i-1)}^{\hbar i} \Sigma_{(NN)}(u)du}$$

as $M \to 0$.

Finally, we use the asymptotic theory developed in [7] to assess the precision of the realized beta $\hat{\beta}$ estimating the latent integrated beta β. Under appropriate regularity conditions that allow for non-stationarity in the time series, the limiting distribution of realized betas are given by

$$\frac{\hat{\beta}_{n,i} - \beta_{n,i}}{\sqrt{\left(\displaystyle\sum_{j=1}^{M} r_{(N),j,i}^2\right)^{-2} \hat{g}_{n,i}}} \xrightarrow{L} N(0,1)$$

as $M \to \infty$, where

$$\hat{g}_{n,i} = \sum_{j=1}^{M} x_{j,i}^2 - \sum_{j=1}^{M-1} x_{j,i} x_{j+1,i}$$

and

$$x_{j,i} = r_{(n),j,i} r_{(n),j,i} - \hat{\beta}_{n,i} r^2_{(n),j,i}$$

Thus a feasible and asymptotically valid $\alpha\%$ confidence interval for $\beta_{n,i}$ is

$$\left[\hat{\beta}_{n,i} - z_{\alpha/2} \sqrt{\left(\sum_{j=1}^{M} r^2_{(N),j,i} \right)^{-2} \hat{g}_{n,i}} \;\; , \;\; \hat{\beta}_{n,i} + z_{\alpha/2} \sqrt{\left(\sum_{j=1}^{M} r^2_{(N),j,i} \right)^{-2} \hat{g}_{n,i}} \right],$$

where $z_{\alpha/2}$ is the critical value of the standard normal distribution.

4.3 Empirical analysis

We set out to investigate the realized monthly betas constructed from intraday returns obtained from TAQ. The Trade and Quote (TAQ) database contains intraday transactions data (trades and quotes) for all securities listed on the New York Stock Exchange (NYSE), American Stock Exchange (AMEX), the Nasdaq National Market System (NMS), and all other U.S. equity exchanges.

Specifically, we study the thirty component stocks of the Dow Jones Industrial Average (DJIA) as of May 2016 with data from January 1, 2010 to December 31, 2014, as detailed in Table 4.1. We take the market return to be the adjusted DJIA. While the DJIA has many excellent attributes, one of its biggest criticisms stems from the fact that it is a price-weighted index. This means that each company is assigned a weighting based on its stock price. In our case, the market-cap-weighted S&P 500 index would be a better proxy for market return; however, the index data is not readily available. Therefore, we adjust DJIA by scaling each price vector by its own maximum as well as taking into account the stock splits. Also, we only consider the stock quotes from 9:30 EST to 16:30 EST on complete trading days during the time period. Given that the DJIA stocks are the most actively traded U.S. equities with a median inter-trade duration at 23.1 seconds, we carry out the analysis at a 5-minute sampling frequency in order to sufficiently reducing the market microstructure effects. There are a total of $M = 1616$ underlying intraday observations for the monthly beta calculations. The main advantage of using the intraday returns is that we obtain less noisy measures for the variance and covariances.

In Figure 4.1 we provide a time series plot of the monthly realized market variance, which is noticeably persistent. Compared to Figure 4.2, which shows a screenshot of the S&P 500 volatility index (VIX) during the same period, our realized market variance displays a similar pattern. It is interesting to note that the first volatility shock is due to the Flash Crash on May 6, 2010, where the DJIA had its biggest intraday point drop of about 9%, although the market quickly recovered afterwards. The second sharp rise in volatility is a result of the stock market fall during August 2011, due to fears of contagion of the European sovereign debt crisis to Spain and Italy, as well as concerns over France's current AAA rating, concerns over the slow economic growth of the United States and its credit rating being downgraded. Severe volatility of stock market index continued for the rest of the year.

In Figure 4.3 and 4.4 we display time series plots of the thirty monthly realized covariances and realized betas. The realized covariances appear highly persistent as the realized variance, whereas the realized betas are less persistent in contrast. This is confirmed by the fact that the median Ljung-Box Q-statistic[1] presented in Table 4.2 for up to sixth-order correlation is 22.93 for the realized covariances, but only 16.13 for the realized beta.[2] The impression of reduced persistence in realized betas relative to realized covariances is also confirmed by the sample autocorrelation functions for the realized market variance, the median realized covariances with the market, and the median realized betas shown in Figure 4.5. Surprisingly, the green market variance correlogram almost completely overlaps with the red median covariances correlogram. This reflects a high degree of dependence in the shape of the covariances' correlograms shown in Figure 4.6.

In contrast, the black median betas correlogram in Figure 4.7 indicates that there is barely any autocorrelation in the median beta time series beyond 5 months. The intuition is that movements of the realized market variance are largely mirrored by those of the realized covariances; as a result, the realized betas, ratios of the realized covariances and the realized market variance, display less autocorrelation. Consequently, there is a lot more heterogeneity among the thirty different beta correlograms shown in Figure 4.7 than the

[1]The Ljung-Box Q-statistic tests the null hypothesis that autocorrelations up to lag k equal zero (that is, the data values are random and independent up to a certain number of lags–in this case six).

[2]The Dickey-Fuller statistics in Table 4.2 indicate that unit roots are not present in the market variance, individual covariances with the market, or individual betas, despite their persistent dynamics.

correlations correlograms shown in Figure 4.6.

So far we have largely ignored the presence of estimation error in the realized betas time series. In Figure 4.8 we plot the point-wise 95% confidence intervals for the underlying (latent) monthly betas using the continuous-record asymptotic theory developed in [7] and detailed in the previous section. The plot directly points to the advantage of using finer samples to improve beta measures since the lower and upper bound almost overlap on each other. We zoom in and take a look at three stocks in particular in Figure 4.9: Boeing, Microsoft, and Du Pont. Compare this to the plot of the 95% confidence bans for the *quarterly* beta with daily sampling in Figure 4.10, we see the significant reduction in measurement error afforded by the use of finer sample intraday data.

It is also evident from Figure 4.9 that there is noticeable positive dependence in the realized daily beta measures. In other words, the high-frequency beta measures importantly complement the results for the betas obtained from the lower frequency daily beta by highlighting the dynamic evolution of individual betas.

4.4 DLM framework

As discussed in the previous section, there is strong evidence of a much lower degree of dependence in the realized betas compared to the realized market variance and the realized covariances with the market return. There is clearly some heterogeneity across the stock betas but a standard short memory autoregressive process with significantly positive serial correlation for each of the individual realized betas appears robust across both estimation horizons and sample lengths.

In light of this, consider the following simple dynamic linear model: denote $y_t = \hat{\beta}_t$ the realized beta and β_t the latent integrated beta.

$$y_t = \beta_t + \nu_t \tag{4.1}$$

$$\beta_t = a + b\beta_{t-1} + \epsilon_t \tag{4.2}$$

where $\nu_t \sim N(0, \sigma_t^2)$ and $\epsilon_t \sim N(0, \tau_t^2)$ are independent, and

$$\sigma_t^2 = \left(\sum_{j=1}^{M} r_{(N),j,t}^2 \right)^{-2} \hat{g}_t.$$

The measurement equation (1) links the observed realized beta to the latent true integrated beta by explicitly introducing a normally distributed error with the asymptotically valid variance σ_t^2 obtained from the continuous-record distribution in [7]. The evolution equation (2) is a standard AR(1) plus noise model with potentially time-varying error variance τ_t^2, which would help alleviate the heteroskedasticity in the realized beta time series. For our relatively short five-year sample, we set τ_t^2 to a constant for simplicity.

We can now obtain samples from the joint posterior of $(a, b, \tau^2, \beta_{0:T})$ using a MCMC scheme together with the Forward Filtering Backward Sampling algorithm (FFBS) for the posterior latent integrated betas. To this end, we build a Gibbs Sampler that iterates through the following steps:

1. Draw (a, b, τ^2, β_0) from

   $$p(a, b, \tau^2, \beta_0|\beta_{1:T}, D_T) \propto p(a|D_T, \dots)p(b|D_T, \dots)p(\tau^2|D_T, \dots)p(\beta_0|D_T, \dots)$$

   where $D_T = \{y_1, \dots, y_T\}$ and $\dots$ represent the other parameters in the joint distribution.

2. Draw $\beta_{1:T}$ from $p(\beta_{1:T}|a, b, \tau^2, \beta_0)$ by first computing forward moments via equations, and then sampling backwards β_t conditional on β_{t+1} and y_t via equation. This step is known as the FFBS algorithm (see, among others, [14, 27]).

Alternatively, Step 1 can be performed by sampling importance resampling, acceptance-rejection algorithm or Metropolis-Hastings-type algorithms. We provide some details on the sampler for completeness here.

4.4.1 Step 1: prior specifications and sufficient statistics

Assume the prior distributions of (a, b, τ^2, β_0) is decomposed into

$$\begin{aligned} \beta_0 &\sim N(m_0, C_0) \\ a &\sim N(a_0, W_0) \\ b &\sim N(b_0, V_0) \\ \tau^2 &\sim IG(n_0/2, n_0 s_0^2/2) \end{aligned}$$

for known hyperparameters $m_0, C_0, a_0, b_0, V_0, W_0, n_0, s_0^2$. It then follows immediately from Bayesian derivations for conditionally conjugate families that

- $(a|b, \tau^2, \beta_0, \beta_{1:T}) \;\sim\; N(a_1, W_1)$ where a_1 and W_1 are given by

$$W_1^{-1} = W_0^{-1} + \frac{n}{\tau^2}, \quad W_1^{-1}a_1 = W_0^{-1}a_0 + \frac{1}{\tau^2}\sum_{t=1}^{T}(\beta_t - b\beta_{t-1})$$

- $(b|a, \tau^2, \beta_0, \beta_{1:T}) \;\sim\; N(b_1, V_1)$ where b_1 and V_1 are given by

$$V_1^{-1} = V_0^{-1} + \frac{1}{\tau^2}\sum_{t=1}^{T}\beta_{t-1}^2, \quad V_1^{-1}b_1 = V_0^{-1}b_0 + \frac{1}{\tau^2}\sum_{t=1}^{T}\beta_{t-1}(\beta_t - a)$$

- $(\tau^2|a, b, \beta_0, \beta_{1:T} \;\sim\; IG(n_1/2, n_1 s_1^2/2)$ where n_1 and s_1^2 are given by

$$n_1 = n_0 + T, \quad n_1 s_1^2 = n_0 s_0^2 + \sum_{t=1}^{T}(\beta_t - a - b\beta_{t-1})^2$$

- $(\beta_0|a, b, \tau^2, \beta_{1:T}) \;\sim\; N(m_1, C_1)$ where m_1 and C_1 are given by

$$C_1^{-1} = C_0 + \frac{b^2}{\tau^2}, \quad C_1^{-1}m_1 = C_0^{-1}m_0 + \frac{b^2}{\tau^2}\beta_1$$

4.4.2 Step 2: FFBS

Conditionally on $\theta = (a, b, \tau^2)$ and assuming the initial distribution $(\beta_0|D_0) \sim N(m_0, C_0)$, we obtain the following densities for $t = 1, \ldots, T$:

$$\text{Propagation density: } (\beta_t|D_{t-1}, \theta) \;\sim\; N(\alpha_t, R_t) \tag{4.3}$$

$$\text{Predictive density: } (y_t|D_{t-1}, \theta) \;\sim\; N(f_t, Q_t) \tag{4.4}$$

$$\text{Filtering density: } (\beta_t|D_t, \theta) \;\sim\; N(m_t, C_t) \tag{4.5}$$

The means and variances for the three densities are provided by the *Kalman recursions*:

$$\alpha_t = a + bm_{t-1} \quad \text{and} \quad R_t = b^2 C_{t-1} + \tau^2 \tag{4.6}$$

$$f_t = \alpha_t \quad \text{and} \quad Q_t = R_t + \sigma_t^2 \tag{4.7}$$

$$m_t = \alpha_t + A_t e_t \quad \text{and} \quad C_t = R_t - R_t A_t \tag{4.8}$$

where $e_t = y_t - f_t$ is the prediction error and $A_t = R_t/Q_t$ is the *Kalman gain*. This completes the forward filtering part (see, in more details, [71]).

Given the conditional independence structure of the model, we have that

$$\begin{aligned} p(\beta_{1:T}|D_T, \theta) &= \prod_{t=1}^{T-1} p(\beta_t|\beta_{t+1:T}, D_T, \theta) p(\beta_T|D_T, \theta) \\ &= \prod_{t=1}^{T-1} p(\beta_t|\beta_{t+1}, D_t, \theta) p(\beta_T|D_T, \theta) \end{aligned}$$

Since the joint density of $(\beta_t, \beta_{t+1}|D_t, \theta)$, we can readily obtain the conditional smoothed density $p(\beta_t|\beta_{t+1}, D_t, \theta) = N(h_t, H_t)$ where

$$h_t = m_t + B_t(\beta_{t+1} - \alpha_{t+1}) \quad \text{and} \quad H_t = C_t - B_t^2 R_{t+1}$$

with $B_t = bC_t/R_{t+1}$. Therefore, the sampling takes place in a backward order: first draw β_T from $p(\beta_T|D_T, \theta)$, then draw β_{T-1} from $p(\beta_{T-1}|\beta_T, D_{T-1}, \theta)$, and keep on going until we get to β_1. Together, $\beta_{1:T}$ is a draw from the joint distribution $p(\beta_{1:T}|D_T, \theta)$.

4.4.3 Empirical analysis

The R code used here is included in Appendix A.

Set the hyperparameters as such: $m_0 = 0, C_0 = 4, a_0 = 0, W_0 = 10, b_0 = 1, V_0 = 10, n_0 = 2, s_0^2 = 2$. We collect 10,000 samples after an initial burn of 50,000 to avoid possible slow convergence of the Markov chain. We also choose the stock that probably best exemplifies the AR(1) structure based on the correlograms in Figure 4.7.

Figure 4.11 shows all 10,000 samples from posterior distribution of a, b, and σ^2 as well as their correlograms and histograms. We see that the Markov chain has converged and there is very little serial correlation in the samples obtained. Figure 4.12 gives the time series plot of median samples from the filtering densities for $\beta_{1:T}$ compared to the actual realization of the betas, while Figure 4.13 plots the 95% confidence bands for the samples. In Figure 4.14, we plot a hundred forecasting paths of β_t for the next 12 months as well as the 95% confidence interval.

4.5 Conclusion

We have assessed the dynamics in realized betas, relative to the dynamics in the underlying market variance and covariances with the market. We find that, unlike the realized variances and covariances fluctuate widely and are highly persistence, the realized beta series, on the other hand, display much less persistence. A critique of the conditional CAPM is that time-varying betas may do more harm than constant betas because it may not be possible to estimate reliably the persistence and predictability in the individual realized betas. This problem, in our case, has been mitigated thanks to the use of high frequency data. Using five-minute intra-day return, we manage to create very narrow 95% confidence intervals for the beta series. We also propose an AR(1) plus noise DLM model for the realized beta, where the measurement error follows a normal distribution centered at zero with asymptotically valid variance given in [7]. This approach helps us obtain samples from filtered and smoothed true underlying beta series and forecast future betas. It is also possible to have a more sophisticated representation for the true underlying beta by utilizing particle filters in order to obtain more accurate short-run forecasts. We look forward to conducting future research along these lines.

4.6 Tables and figures

Table 4.1: The Dow Jones Thirty

Ticker	Company Name	Industry
AAPL	Apple	Consumer electronics
AXP	American Express	Consumer finance
BA	Boeing	Aerospace and defense
CAT	Caterpillar	Construction and mining equipment
CSCO	Cisco	Computer networking
CVX	Chevron	Oil & gas
DD	Du Pont	Chemical industry
DIS	Walt Disney	Broadcasting and entertainment
GE	General Electric	Conglomerate
GS	Goldman Sachs	Banking, Financial services
HD	Home Depot	Home improvement retailer
IBM	IBM	Computers and technology
INTC	Intel	Semiconductors
JNJ	Johnson & Johnson	Pharmaceuticals
JPM	JPMorgan Chase	Banking
KO	Coca-Cola	Beverages
MCD	McDonald's	Fast food
MMM	3M	Conglomerate
MRK	Merck	Pharmaceuticals
MSFT	Microsoft	Software
NKE	Nike	Apparel
PFE	Pfizer	Pharmaceuticals
PG	Procter & Gamble	Consumer goods
T	AT&T	Telecommunication
UNH	UnitedHealth Group	Managed health care
UTX	United Technologies	Conglomerate
V	Visa	Consumer banking
VZ	Verizon	Telecommunication
WMT	Wal-Mart	Retail
XOM	ExxonMobil	Oil & gas

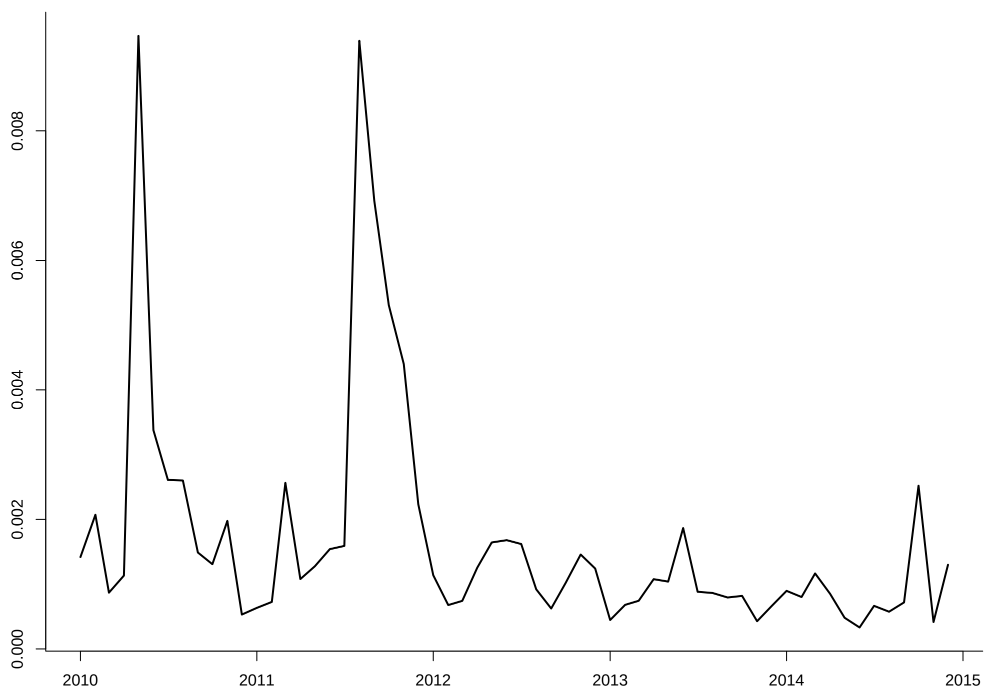
Notes: The table summarizes company names, tickers, and industries of the thirty stocks in the Dow Jones Thirty as of May 2016.

Table 4.2: The Dynamics of Monthly Realized Market Variance, Covariances, and Betas

	Q	ADF(1)	ADF(2)	ADF(3)	ADF(4)
Market Var.	23.39	-3.97	-3.66	-4.16	-3.49
Covariances					
Min	8.12	-4.82	-4.51	-4.79	-4.31
Median	22.93	-4.04	-3.68	-4.21	-3.48
Max	41.58	-3.23	-3.34	-3.59	-2.95
β					
Min	3.22	-6.33	-5.19	-4.43	-4.14
Median	16.13	-4.02	-3.24	-3.13	-2.71
Max	131.34	-2.32	-1.86	-1.42	-1.60

Notes: Q denotes the Ljung-Box portmanteau statistic for up to sixth-order autocorrelation, and ADF(i) denotes the augmented Dickey-Fuller unit root test with i augmentation lags. The sample covers the period from January 2010 through December 2014, for a total of 60 observations. We calculate the realized monthly realized market variances from five-minute returns.

Figure 4.1: Time Series Plot of Monthly Realized Market Variance

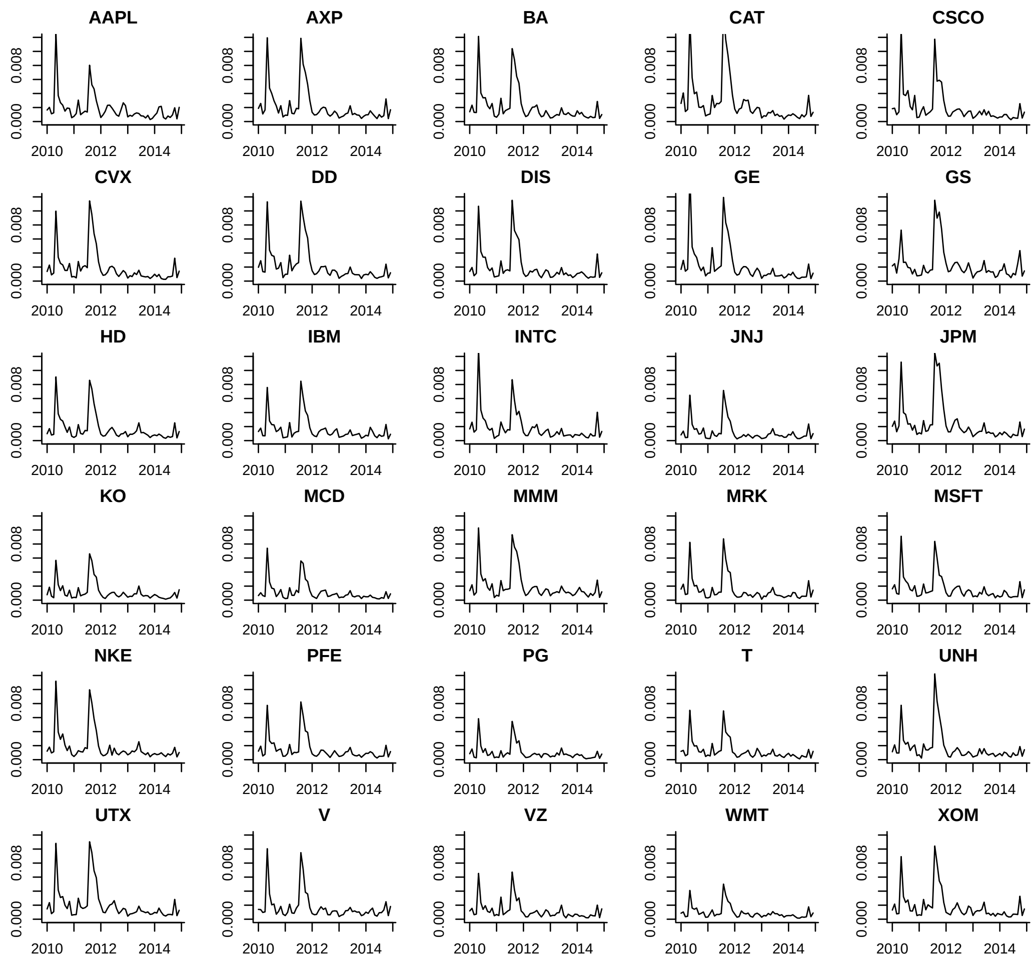


Notes: The figure shows the times series of monthly realized market variance. The sample covers the period from January 2010 through December 2014, for a total of 60 observations. We calculate the realized monthly realized market variances from five-minute returns.

Figure 4.2: Volatility Index (VIX) of S&P 500

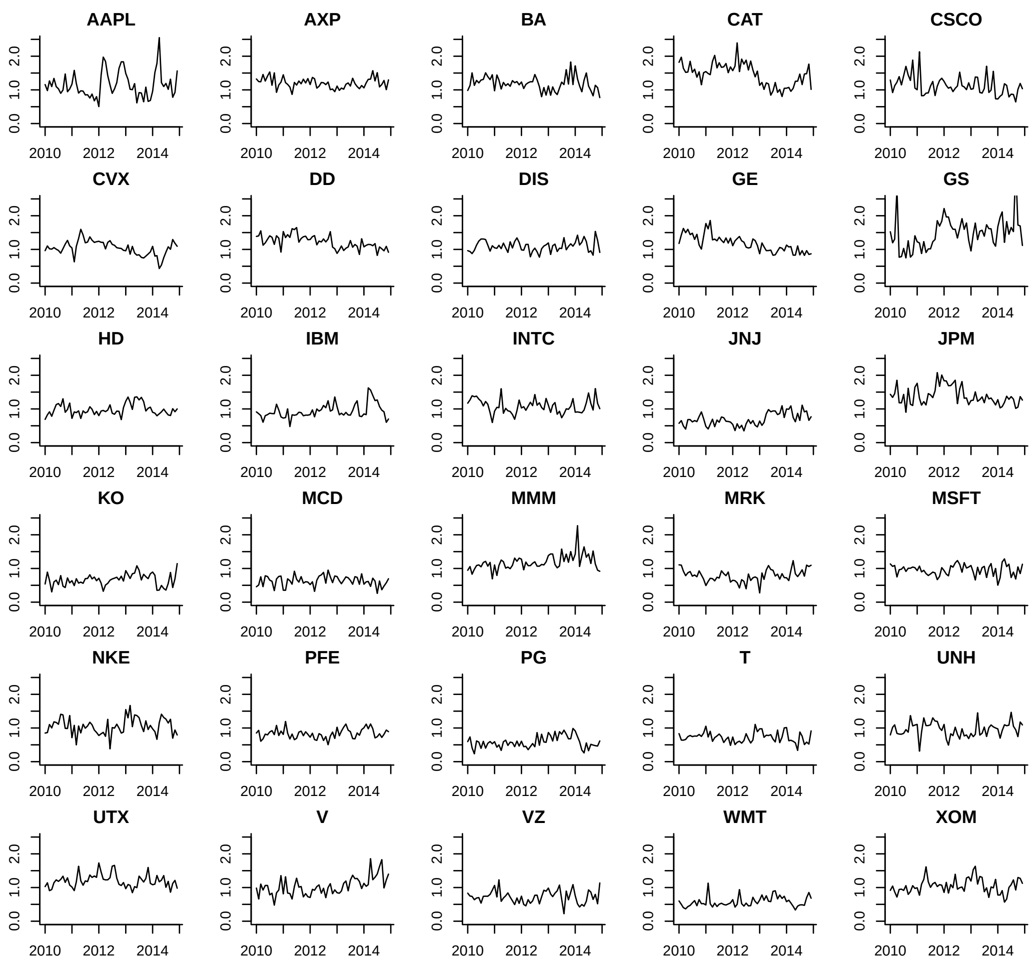
Notes: The figure shows a screenshot of the volatility index (VIX) of S&P 500 from January 2010 through December 2014 from Yahoo Finance.

Figure 4.3: Time Series Plots of Monthly Realized Covariances



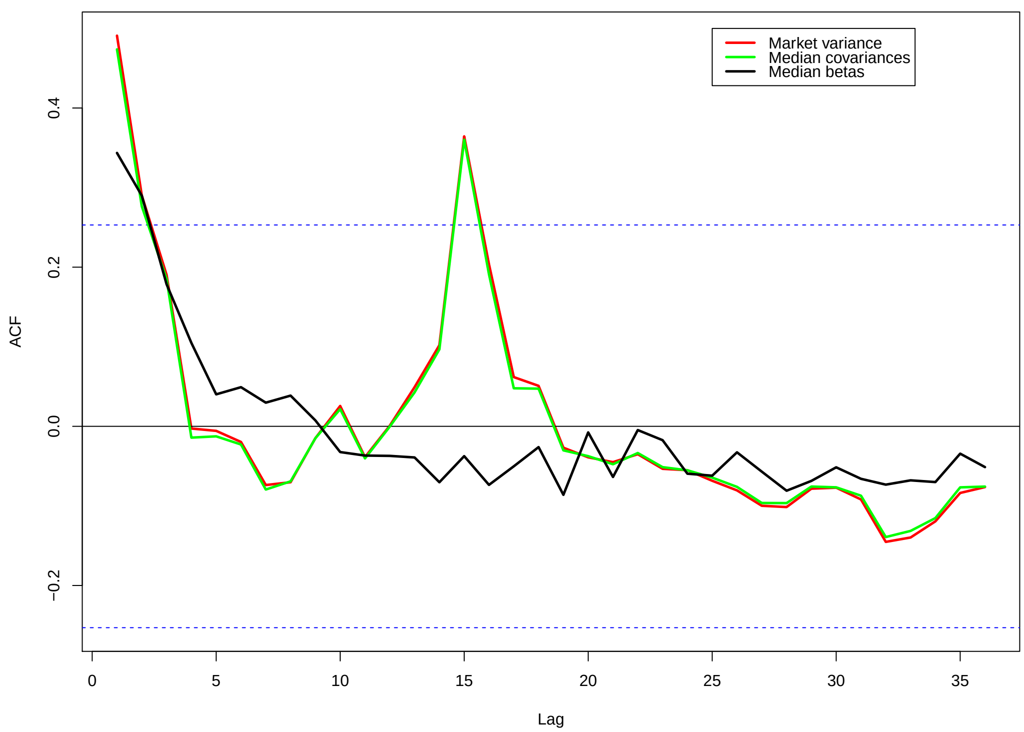
Notes: The figure shows the times series of monthly realized covariances. The sample covers the period from January 2010 through December 2014, for a total of 60 observations. We calculate the realized monthly realized market variances from five-minute returns.

Figure 4.4: Time Series Plots of Monthly Realized Betas



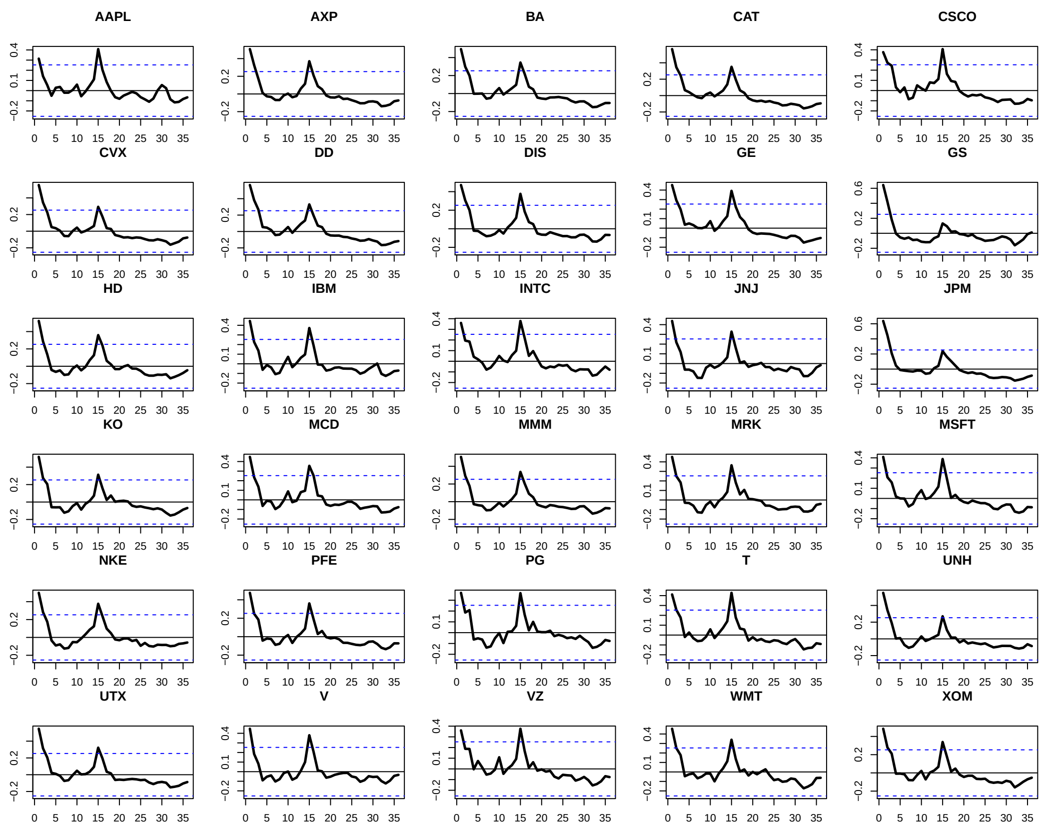
Notes: The figure shows the times series of monthly realized betas. The sample covers the period from January 2010 through December 2014, for a total of 60 observations. We calculate the realized monthly realized market variances from five-minute returns.

Figure 4.5: Sample Autocorrelations of Monthly Realized Market Variance, Median Sample Autocorrelations of Monthly Realized Covariances, and Median Sample Autocorrelations of Monthly Realized Betas



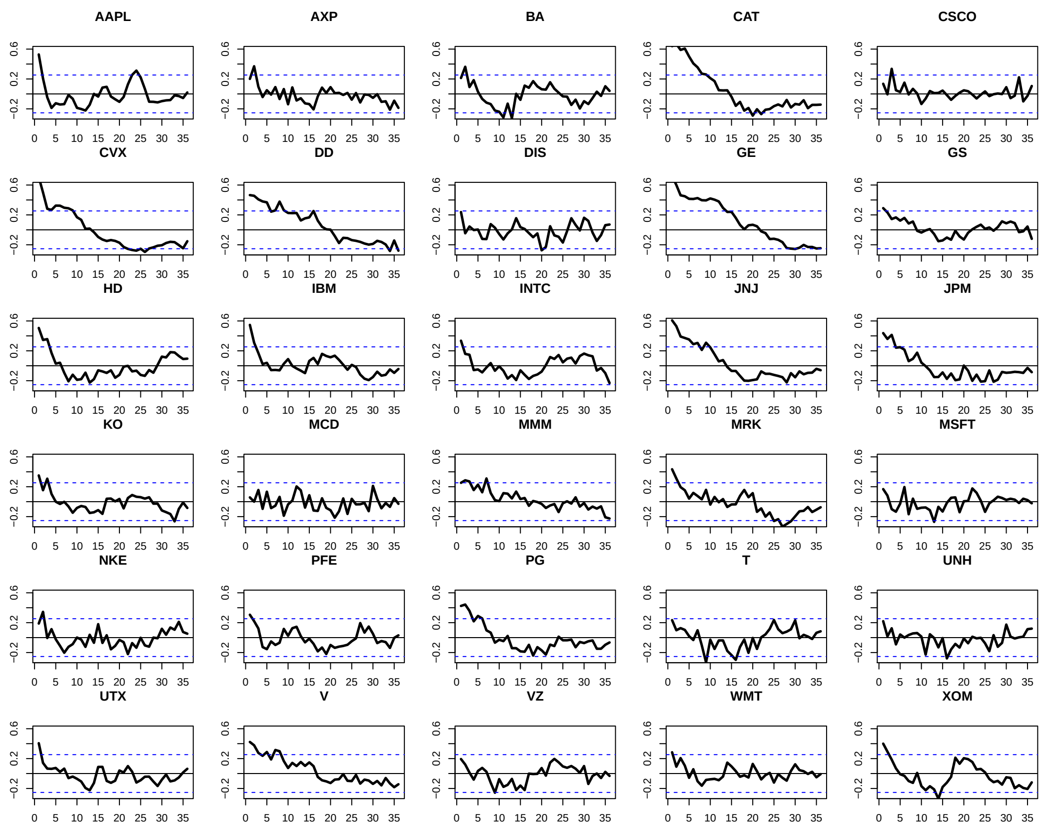
Notes: The figure shows the first 36 sample autocorrelations of the monthly realized market variance, the medians across individual stocks of the first 36 sample autocorrelations of monthly realized covariances, and the medians across individual stocks of the first 36 sample autocorrelations of monthly realized betas. The dashed lines denote Bartlett's approximate 95% confidence band in the white noise case. The sample covers the period from January 2010 through December 2014, for a total of 60 observations. We calculate the realized monthly realized market variances from five-minute returns.

Figure 4.6: Sample Autocorrelations of Monthly Realized Covariances



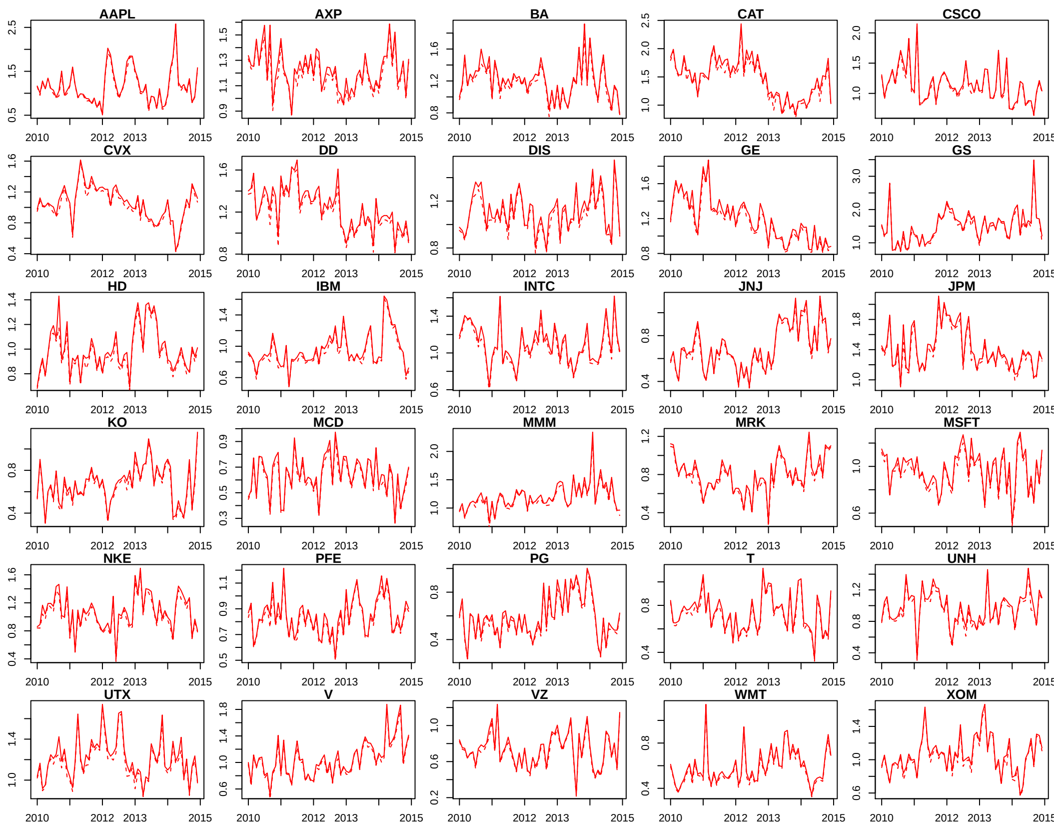
Notes: The figure shows the first 36 sample autocorrelations of the monthly covariances. The dashed lines denote Bartlett's approximate 95% confidence band in the white noise case. The sample covers the period from January 2010 through December 2014, for a total of 60 observations. We calculate the realized monthly realized market variances from five-minute returns.

Figure 4.7: Sample Autocorrelations of Monthly Realized Betas



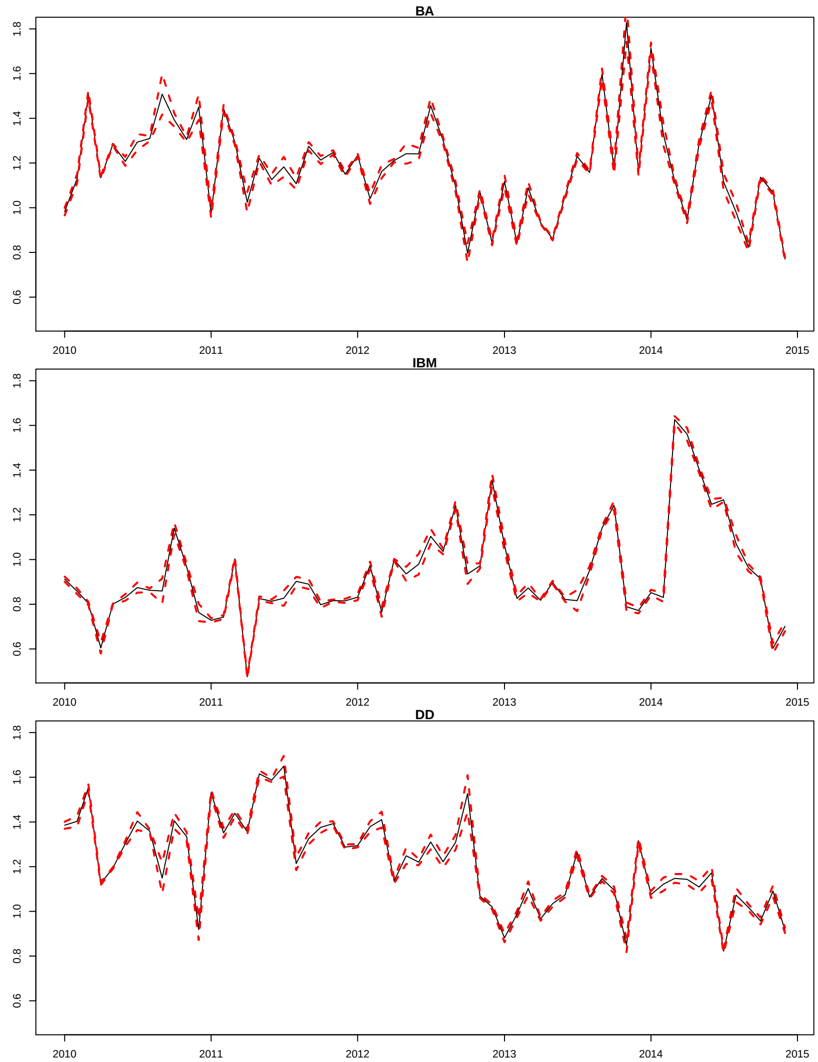
Notes: The figure shows the first 36 sample autocorrelations of the monthly betas. The dashed lines denote Bartlett's approximate 95% confidence band in the white noise case. The sample covers the period from January 2010 through December 2014, for a total of 60 observations. We calculate the realized monthly realized market variances from five-minute returns.

Figure 4.8: 95% Confidence Intervals for Monthly Beta, Five-minute Sampling



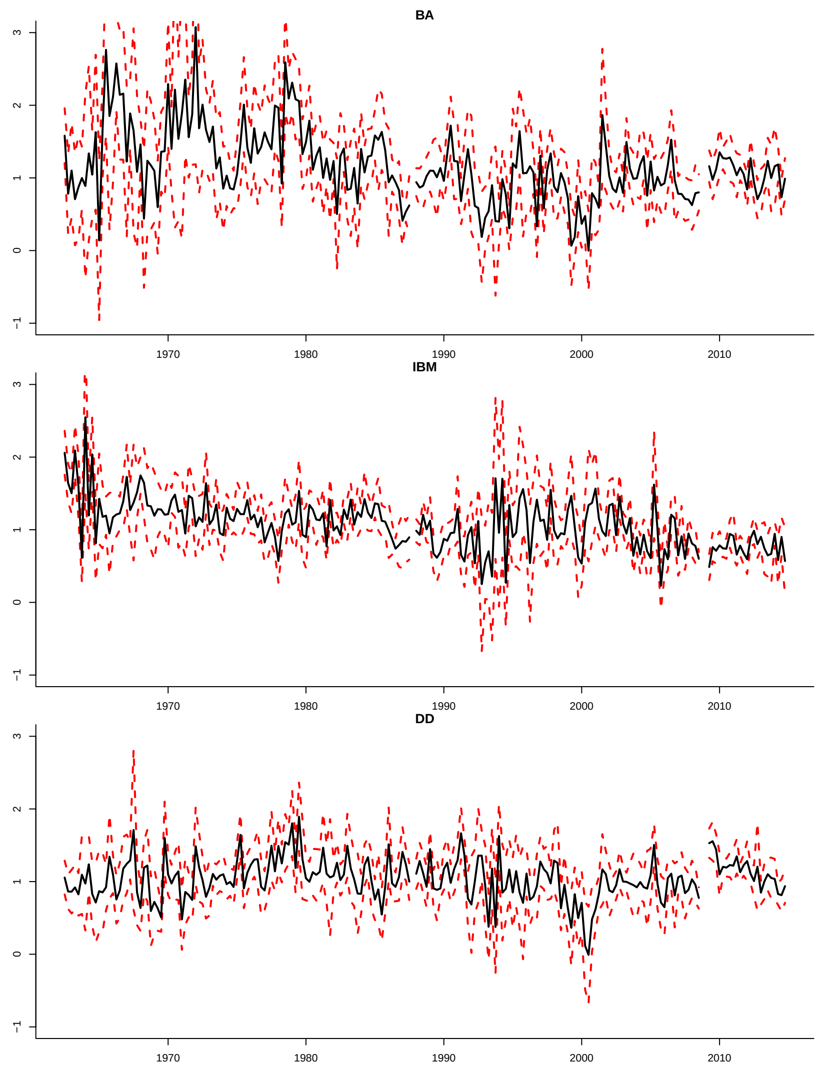
Notes: The figure shows the times series of 95% Confidence Intervals for the underlying monthly integrated betas, calculated using the results in [7]. The sample covers the period from January 2010 through December 2014, for a total of 60 observations. We calculate the monthly realized beta from five-minute returns.

Figure 4.9: 95% Confidence Intervals for Monthly Beta of BA/IBM/DD, Five-minute Sampling



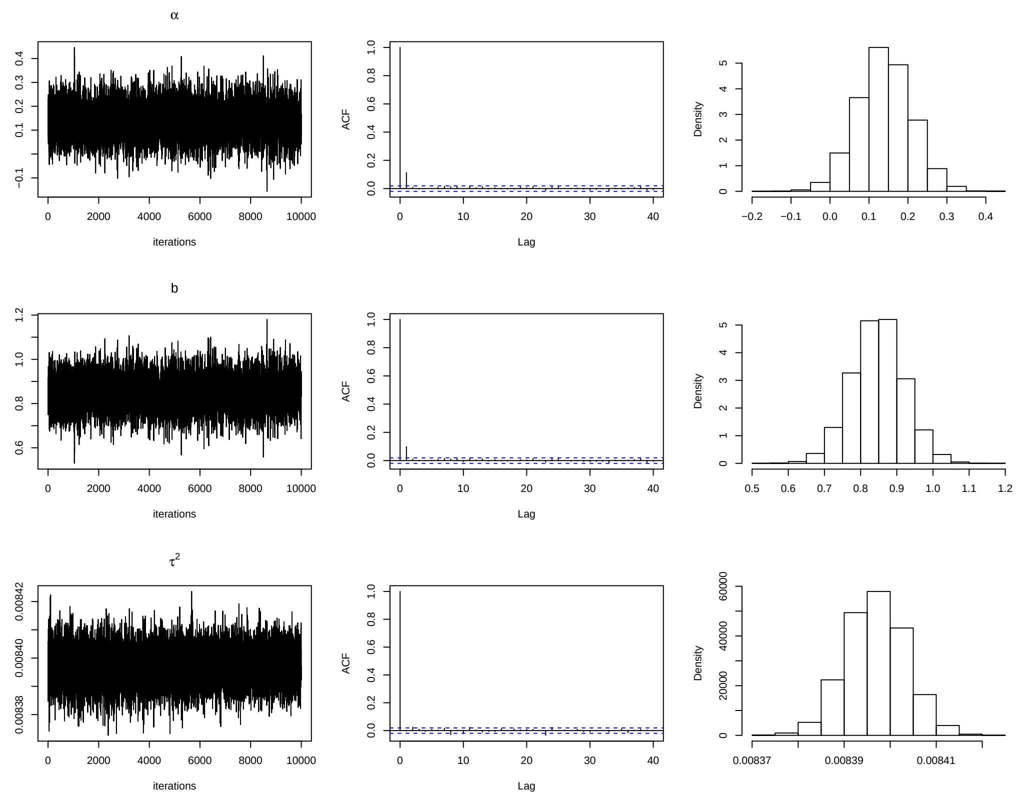
Notes: The figure shows the times series of 95% Confidence Intervals for the underlying monthly integrated betas of Boeing, IBM, and Du Pont, calculated using the results in [7]. The sample covers the period from January 2010 through December 2014, for a total of 60 observations. We calculate the **monthly** realized beta from **five-minute** returns.

Figure 4.10: 95% Confidence Intervals for Quarterly Beta of BA/IBM/DD, Daily Sampling



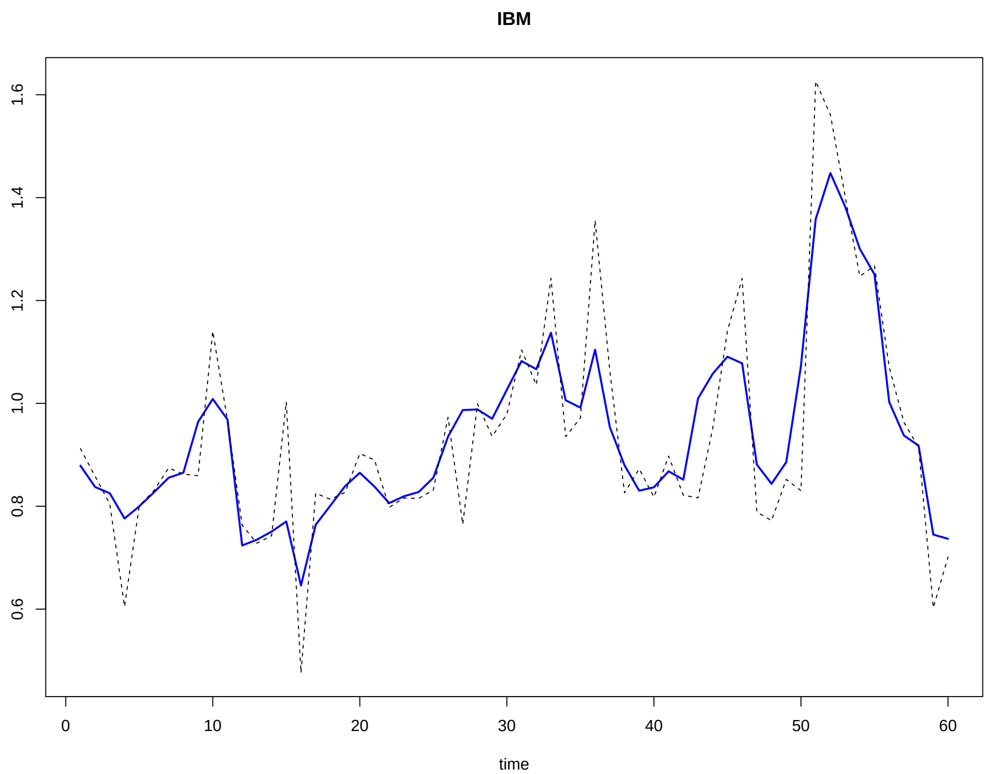
Notes: The figure shows the times series of 95% Confidence Intervals for the underlying monthly integrated betas of Boeing, IBM, and Du Pont, calculated using the results in [7]. The sample covers the period from July 1962 through December 2014. We calculate the **quarterly** realized beta from **daily** returns.

Figure 4.11: Samples from Posterior Distribution of a, b, and σ^2 for Monthly Realized Beta of IBM



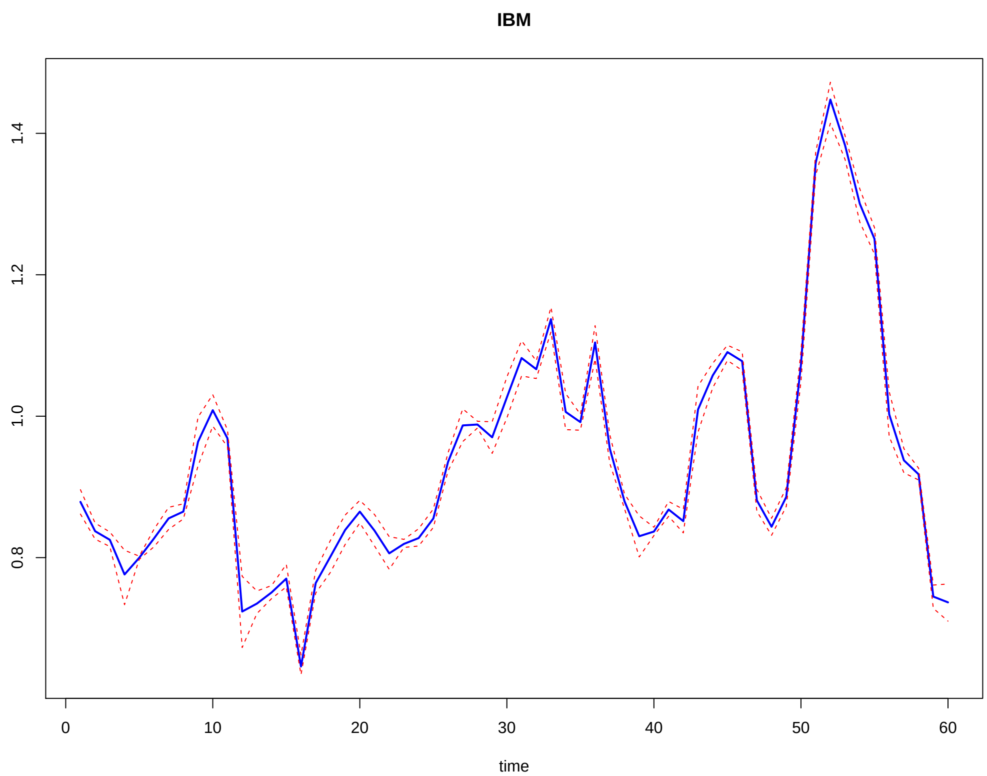
Notes: The figure shows samples from posterior distribution of a, b, and σ^2 for monthly realized betas of IBM. The sample covers the period from January 2010 through December 2014, for a total of 60 observations. We calculate the monthly realized beta from five-minute returns.

Figure 4.12: Time Series Plot of Median Smoothed Samples and Actual Realization for Monthly Realized Beta for IBM



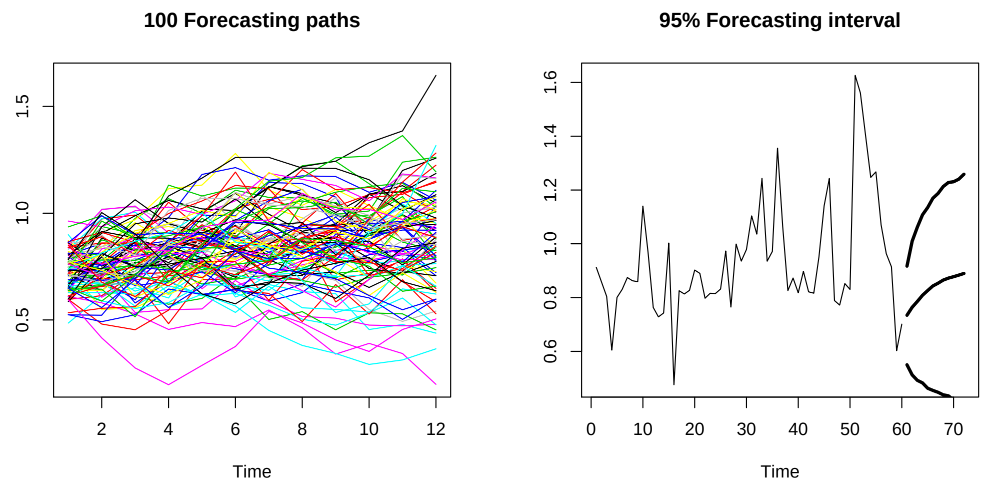
Notes: The figure shows the time series plot of median smoothed samples (blue) and the actual realization (black dotted) for monthly realized betas of IBM. The sample covers the period from January 2010 through December 2014, for a total of 60 observations. We calculate the monthly realized beta from five-minute returns.

Figure 4.13: Time Series Plot of Median Smoothed Samples and 95% Confidence Bands for Monthly Realized Beta for IBM



Notes: The figure shows the time series plot of median smoothed samples (blue) and the 95% Confidence band (red dotted) for monthly realized betas of IBM. The sample covers the period from January 2010 through December 2014, for a total of 60 observations. We calculate the monthly realized beta from five-minute returns.

Figure 4.14: 100 Forecasting Paths for the Next 12 Months of Monthly Realized Beta for IBM and their 95% Confidence Interval



Notes: The figure shows 100 forecasting paths for the next 12 months of monthly realized betas for IBM and their 95% confidence interval.

Appendix

R code for the DLM model

```
########################################################################
# DLM: AR(1) PLUS NOISE MODEL using FFBS
#      Observation equation:
#      y(t)|beta(t) ~ N(beta(t),sig2_t)
#      Evolution equation:
#      beta(t)|beta(t-1),theta ~ N(alpha+b*beta(t-1),tau2)
# where theta=(alpha,b,tau2), beta(0) ~ N(m0,C0), alpha ~ N(alpha0,W0),
# b ~ N(b0,V0), tau2 ~ IG(n0/2,n0/2*s0), and sig2_t are known.
########################################################################
# FFBS    - Sampling from p(beta(1:T)|y(1:T),theta).
# MCMC    - Sampling from p(beta(1:T),theta|y(1:T)) via Gibbs sampler:
#                p(beta(1:T)|theta,y(1:T)) via FFBS.
#                p(theta|beta(0:T),y(1:T)) via Bayesian updates
########################################################################

FFBS <- function(y,alpha,b,sig2,tau2,m0,C0,M){
  # number of samples
  n <- length(y)
  ##### Forward Filtering #####
  mf <- rep(0,n)
```

```
Cf <- rep(0,n)
# mean and variance of propagation density
a <- rep(0,n)
a[1] <- alpha + b * m0
R <- rep(0,n)
R[1] <- b^2 * C0 + tau2
# auxiliary parameter
B <- rep(0,n-1)
# mean of predictive density = a
# variance of predictive density
Q <- R[1] + sig2[1]
l <- dnorm(y[1],a[1],sqrt(Q),log=TRUE)
# Kalman gain
A <- R[1]/Q
# mean and variance of filtering density
mf[1] <- a[1] + A * (y[1] - a[1])
Cf[1] <- A * sig2[1]
for (t in 2:n){
  R[t] <- b^2 * Cf[t-1] + sig2[t]
  B[t-1] <- b * Cf[t-1] / R[t]
  a[t] <- alpha  + b * mf[t-1]
  Q <- R[t] + sig2[t]
  l <- l + dnorm(y[t],a[t],sqrt(Q),log=TRUE)
```

```
    A <- R[t]/Q
    mf[t] <- a[t] + A * (y[t]-a[t])
    Cf[t] <- A * sig2[t]
  }
  ##### Backward Sampling #####
  beta <- matrix(0,M,n)
  beta[,n] <- rnorm(M,mf[n],sqrt(Cf[n]))
  for (t in (n-1):1){
    # mean and variance of conditional smoothed density
    mb <- mf[t] + B[t]*(beta[,t+1]-a[t+1])
    Cb <- Cf[t] - B[t]^2 * R[t+1]
    beta[,t] <- rnorm(M,mb,sqrt(Cb))
  }
  return(list(beta=beta,l=l))
}

MCMC <- function(y,alpha,b,sig2,tau2,m0,C0,alpha0,W0,b0,
V0,n0,s0,burn,niter){
  n <- length(y)
  W <- W0
  V <- V0
  sig2 <- sig2
  tau2 <- tau2
```

```
n1 <- n0
s1 <- s0
draw <- c(alpha,b,tau2)
draws <- matrix(0,niter,n+3)
for (iter in 1:niter){
  # sample beta(1:n)vrom FFBS
  beta <- (FFBS(y,draw[1],draw[2],sig2,draw[3],m0,C0,1)$beta)[1,]
  # sampling beta(0)
  C1 <- 1/(1/C0+b^2/tau2)
  m1 <- C1*(m0/C0+(beta[1]-alpha)*b/tau2)
  beta0 <- rnorm(1,m1,sqrt(C1))
  beta1 <- c(beta0,beta[1:n-1])
  # sampling (alpha,b) from their conditional
  BBeta <- cbind(1,beta1)
  A0 <- tau2*diag(c(1/W0,1/V0),2)
  v <- solve(A0+t(BBeta)%*%BBeta)
  m <- v%*%(t(BBeta)%*% beta+A0%*%c(alpha0,b0))
  ab <- m+t(chol(v))%*%rnorm(2,0,sqrt(tau2))
  alpha <- ab[1]
  b <- ab[2]
  # sample tau2 from its conditional
  n1_new <- n1 + n
  s1 <- (n1*s1+sum((beta-alpha-b*beta1)^2))/n1_new
```

```
    n1 <- n1_new
    tau2 <- 1/rgamma(1,n1/2,n1*s1/2)
    draw <- c(alpha,b,tau2)
    draws[iter,] <- c(beta,draw)
  }
  return(draws[(burn+1):niter,])
}

# 95% confidence band
q025 <- function(x){quantile(x,0.025)}
q975 <- function(x){quantile(x,0.975)}

## Simulating DLM
ex <- function(stock,M){
  # Hyperparameters
  m0 <- 0
  C0 <- 4
  alpha0 <- 0
  b0 <- 1
  W0 <- 10
  V0 <- 10
  n0 <- 2
  s0 <- 2
```

```
y <- beta_list[[stock]]
n <- length(y)
sig2 <- (betaSD_list[[stock]])^2
if (n!=length(sig2)) stop("sig2 and y have different lengths")
tau2 <- 1/(n0*s0/4)
burn <- 50000
niter <- burn + M
draws <- MCMC(y,alpha0,b0,sig2,tau2,m0,C0,alpha0,
 W0,b0,V0,n0,s0,burn,niter)
betas <- draws[,1:n]
lbeta <- apply(betas,2,q025)
mbeta <- apply(betas,2,median)
ubeta <- apply(betas,2,q975)
draw <- draws[,(n+1):(n+3)]
names <- c(expression(alpha),"b",expression(tau^2))
par(mfrow=c(3,3))
for (i in 1:3){
  ts.plot(draw[,i],xlab="iterations",ylab="",main=names[i])
  acf(draw[,i],main="")
  hist(draw[,i],prob=TRUE,xlab="",main="")
}
par(mfrow=c(1,1))
ts.plot(mbeta,xlab="time",ylab="",lwd=2,
```

```
ylim=range(y),col=4,main=stock)
  lines(y,lwd=1,lty=2)
  legend(0,2.5,c("Posterior median","Actual realization"),
  lty=c(1,2),lwd=c(2.5,2.5),col=c("blue","black"))
  par(mfrow=c(1,1))
  ts.plot(mbeta,xlab="time",ylab="",lwd=2,
ylim=range(lbeta,ubeta),col=4,main=stock)
  lines(lbeta,col=2,lty=2)
  lines(ubeta,col=2,lty=2)
  # k-steps ahead forecasting
  alpha <- draw[,1]
  b <- draw[,2]
  tau <- sqrt(draw[,3])
  k <- 12
  yf <- matrix(0,M,k)
  yf[,1] <- rnorm(M,alpha+b*y[n],tau)
  for (i in 2:k){
    yf[,i] = rnorm(M,alpha+b*yf[,i-1],tau)
  }
  myf = apply(yf,2,median)
  lyf = apply(yf,2,q025)
  uyf = apply(yf,2,q975)
  ind = sample(1:M,size=100,replace=FALSE,prob=rep(1/M,M))
```

```
  par(mfrow=c(1,2))
  ts.plot(t(yf[ind,]),col=1:20,ylim=range(yf))
  title("100 Forecasting paths")
  plot(y,xlim=c(1,n+k),xlab="Time",ylab="",main="",
type="l",ylim=range(yf))
  lines((n+1):(n+k),lyf,col=1,lwd=3)
  lines((n+1):(n+k),myf,col=1,lwd=3)
  lines((n+1):(n+k),uyf,col=1,lwd=3)
  title("95% Forecasting interval")
}
```

Bibliography

[1] A. B. Abel. Asset prices under habit formation and catching up with the joneses. *The American Economic Review*, 80(2):38–42, 1990.

[2] V. Agarwal, N. D. Daniel, and N. Y. Naik. Flows, performance, and managerial incentives in hedge funds. In *EFA 2003 Annual Conference Paper No. 501*, 2003.

[3] D. Aliprantis and O. Burkinshaw. *Principles of Real Analysis*. Academic Press, 3rd edition, 1998.

[4] B. Angoshtari, T. Zariphopoulou, and X. Y. Zhou. Predictable forward performance processes: The binomial case. Submitted for publication, November 2016.

[5] A. S. Azar. Measuring relative risk aversion. *Applied Financial Economics Letters*, 2(5):341–345, 2006.

[6] N. Barberis and M. Huang. Stocks as lotteries: The implications of probability weighting for security prices. *The American Economic Review*, 98(5):2066–2100, 2008.

[7] O. E. Barndorff-Nielsen and N. Shephard. Econometric analysis of realized covariation: High frequency based covariance, regression, and correlation in financial economics. *Econometrica*, 72(3):885–925, 2004.

[8] S. Basak and D. Makarov. Competition among portfolio managers and asset specialization. Paris December 2014 Finance Meeting EUROFIDAI - AFFI Paper, September 2015.

[9] S. Basak, A. Pavlova, and A. Shapiro. Optimal asset allocation and risk shifting in money management. *The Review of Financial Studies*, 20(5):1583–1621, 2007.

[10] F. Berrier, C. Rogers, and M. Tehranchi. A characterization of forward utility functions. Prerprint, May 2009.

[11] B. Bian and H. Zheng. Turnpike property and convergence rate for an investment model with general utility functions. *Journal of Economic Dynamics and Control*, 51:28, 2015.

[12] P. Billinglsey. *Probability and measure*. J. Wiley & Sons, New York, 3rd edition, 1995.

[13] J. Y. Campbell, A. W. Lo, and . MacKinlay, Archie C. *The econometrics of financial markets*. Princeton University Press, Princeton, N.J, 1997.

[14] C. K. Carter and R. Kohn. On gibbs sampling for state space models. *Biometrika*, 81(3):541–553, 1994.

[15] R. Chetty. A new method of estimating risk aversion. *The American Economic Review*, 96(5):1821–1834, 2006.

[16] J. Chevalier and G. Ellison. Career concerns of mutual fund managers. *The Quarterly Journal of Economics*, 114(2):389–432, 1999.

[17] J. H. Cochrane. *Asset Pricing*. Princeton University Press, 2001.

[18] R. Corless, G. Gonnet, D. Hare, and D. Jeffrey. On the lambert w function. *Adv. Comput. Math.*, 5(4):329–359, 1996.

[19] J. D. Coval and T. J. Moskowitz. Home bias at home: Local equity preference in domestic portfolios. *The Journal of Finance*, 54(6):2045–2073, 1999.

[20] J. C. Cox and C.-f. Huang. A continuous-time portfolio turnpike theorem. *Journal of Economic Dynamics and Control*, 16(3–4):491–507, 1992.

[21] D. Cuoco and R. Kaniel. Equilibrium prices in the presence of delegated portfolio management. *Journal of Financial Economics*, 101(2):264–296, 2011.

[22] P. H. Dybvig, L. C. G. Rogers, and K. Back. Portfolio turnpikes. *The Review of Financial Studies*, 12(1):pp. 165–195, 1999.

[23] R. C. Entringer. Functions and inverses of asymptotic functions. *The American Mathematical Monthly*, 74(9):1095–1097, 1967.

[24] P. Erdos. On the law of the iterated logarithm. *Annals of Mathematics*, 43(3):419–436, 1942.

[25] E. Fama. *Foundations of finance: portfolio decisions and securities prices*. Basic Books, New York, 1976.

[26] W. Feller. The general form of the so-called law of the iterated logarithm. *Transactions of the American Mathematical Society*, 54(3):373–402, 1943.

[27] S. Frühwirth-Schnatter. Data augmentation and dynamic linear models. *Journal of Time Series Analysis*, 15:183–202, 1994.

[28] S. Gallaher, R. Kaniel, and L. T. Starks. Madison avenue meets wall street: Mutual fund families, competition and advertising. Working paper, January 2006.

[29] T. Geng and T. Zariphopoulou. On the asymptotic properties of the optimal wealth and portfolio weight processes under time-monotone forward performance criteria in itô-diffusion markets. Preprint, 2017.

[30] T. Geng and T. Zariphopoulou. Temporal and spatial turnpike-type results under forward time-monotone performance criteria. Preprint, 2017.

[31] M. Getmansky, B. Liang, C. Schwarz, and R. Wermers. Share restrictions and investor flows in the hedge fund industry. Preprint, November 2015.

[32] J.-P. Gómez, R. Priestley, and F. Zapatero. Implications of keeping-up-with-the-joneses behavior for the equilibrium cross section of stock returns: International evidence. *The Journal of Finance*, 64(6):2703–2737, 2009.

[33] P. Guasoni, C. Kardaras, S. Robertson, and H. Xing. Abstract, classic, and explicit turnpikes. *Finance and stochastics*, 18(1):75–114, 2014.

[34] C.-f. Huang and R. H. Litzenberger. *Foundations for financial economics*. North-Holland, New York, 1988.

[35] C.-f. Huang and T. Zariphopoulou. Turnpike behavior of long-term investments. *Finance and stochastics*, 3(1):15–34, 1999.

[36] G. Huberman and S. Ross. Portfolio turnpike theorems, risk aversion and regularly varying utility functions. *Econometrica*, 51(5):1345–1361, 1983.

[37] .-. Itō, Kiyosi and H. P. McKean. *Diffusion processes and their sample paths*, volume Bd. 125;Bd. 125.;. Springer-Verlag, New York;Berlin;, 1965.

[38] K. Janeček. What is a realistic aversion to risk for real-world individual investors? http://www.sba21.com/personal/RiskAversion.pdf, 2004.

[39] X. Jin. Consumption and portfolio turnpike theorems in a continuous-time finance model. *Journal of Economic Dynamics and Control*, 22(7):1001–1026, 1998.

[40] M. Kacperczyk, C. Sialm, and L. Zheng. On the industry concentration of actively managed equity mutual funds. *The Journal of Finance*, 60(4):1983–2011, 2005.

[41] S. Källblad, J. Obłój, and T. Zariphopoulou. Dynamically consistent investment under model uncertainty: the robust forward criteria. To appear in Finance and Stochastics.

[42] S. Kallblad and T. Zariphopoulou. On the black's equation for the risk tolerance function. *Preprint*, 2017.

[43] R. Kaniel, S. Tompaidis, and T. Zhou. Impact of managerial commitment on risk taking with dynamic fund flows. Preprint, April 2016.

[44] N. E. Karoui and M. Mrad. An exact connection between tow solvable sdes and a nonlinear utility stochastic pde. *SIAM Journal on Financial Mathematics*, 4(1):697–736, 2014.

[45] J. L. Kelly. A new interpretation of information rate. *The Bell System Technical Journal*, 35(4):917–926, 1956.

[46] R. S. J. Koijen. The crosssection of managerial ability, incentives, and risk preferences. *The Journal of Finance*, 69(3):1051–1098, 2014.

[47] H. Kunita. *Stochastic flows and stochastic differential equations*, volume 24. Cambridge University Press, 1990.

[48] P. Lévy. Sur un théorème de m. khintchine. *Bulletin des Sciences Mathématiques*, 55(2):145–160, 1931.

[49] G. Liang and T. Zariphopoulou. Representation of homothetic forward performance processes in stochastic factor models via ergodic and infinite horizon bsde. submitted for publication, November 2016.

[50] J. Lintner. Security prices, risk, and maximal gains from diversification. *The Journal of Finance*, 20(4):587–615, 1965.

[51] J. Lintner. The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets: A reply. *The Review of Economics and Statistics*, 51(2):222–224, 1965.

[52] R. C. Merton. A simple model of capital market equilibrium with incomplete information. *The Journal of Finance*, 42(3):483–510, 1987.

[53] T. Mitton and K. Vorkink. Equilibrium underdiversification and the preference for skewness. *The Review of Financial Studies*, 20(4):1255–1288, 2007.

[54] P. Monin. On a dynamic adaptation of the distribution builder approach to investment decisions. *Quantitative Finance*, 14(5):749–760, 2014.

[55] M. Musiela and T. Zariphopoulou. Investments and forward utilities. Technical report, University of Texas at Austin, 2006.

[56] M. Musiela and T. Zariphopoulou. *Derivative pricing, investment management and the term structure of exponential utilities: The case of binomial model*. Princeton University Press, 2009.

[57] M. Musiela and T. Zariphopoulou. Portfolio choice under dynamic investment performance criteria. *Quantitative Finance*, 9(2):161–170, 2009.

[58] M. Musiela and T. Zariphopoulou. Portfolio choice under space-time monotone performance criteria. *SIAM Journal on Financial Mathematics*, 1(1):326–365, 2010.

[59] M. Musiela and T. Zariphopoulou. Stochastic partial differential equations and portfolio choice. *Contemporary Quantitative Finance*, pages 195–215, 2010.

[60] S. Nadtochiy and M. Tehranchi. Optimal investment for all time horizons and martin boundary of space-time diffusions. *Mathematical Finance*, published online.

[61] S. Nadtochiy and T. Zariphopoulou. A class of homothetic forward investment performance process with non-zero volatility. *Inspired by Finance: The Musiela Festschrift*, pages 475–505, 2014.

[62] J. A. Parker. Consumption risk and expected stock returns. *The American Economic Review*, 93(2):376–382, 2003.

[63] H. Robbins and D. Siegmund. Statistical tests of power one and the integral representation of solutions of certain parabolic differential equations. *Bull. Inst. Math. Acad. Sinica (Taipei)*, 1:93–120, 1973.

[64] W. Sharpe. A simplified model for portfolio analysis. *Management Science*, 9(2):277–293, 1963.

[65] W. Sharpe, D. Goldstein, and P. Blythe. The distribution builder: A tool for inferring investor preferences. 2000.

[66] M. Shkolnikov, R. Sircar, and T. Zariphopoulou. Asymptotic analysis of forward performance processes in incomplete markets and their ill-posed hjb equations. *SIAM Journal on Financial Mathematics*, 7(1):588–618, 2016.

[67] E. R. Sirri and P. Tufano. Costly search and mutual fund flows. *The Journal of Finance*, 53(5):1589–1622, 1998.

[68] R. Uppal and T. Wang. Model misspecification and underdiversification. *The Journal of Finance*, 58(6):2465–2486, 2003.

[69] J. H. van Binsbergen, M. W. Brandt, and R. S. J. Koijen. Optimal decentralized investment management. *The Journal of Finance*, 63(4):1849–1895, 2008.

[70] W. Wagner. Systemic liquidation risk and the diversity-diversification trade-off. *The Journal of Finance*, 66(4):1141–1175, 2011.

[71] M. West and J. Harrison. *Bayesian forecasting and dynamic models.* Springer, New York, 2nd edition, 1999.

[72] D. V. Widder. *The Laplace Transform.* Princeton University Press, 1941.

[73] G. Žitković. A dual characterization of self-generation and exponential forward performances. *Annals of Applied Probability*, 19(6):2176–2210, 2009.